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Variational analysis of some questions in dynamical system and biological system

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ABSTRACT

This paper studies the variational problems of some dynamical system. By using the methods of Euler equation and Legendre conditions and some computional techniques in mechanics and differential equations, this paper computes the extrema of some systems and discuss some future applications in symplectic dynamical systems and biological mathematical problems.

Key words: variational methods; Euler equation; biological mathematical problems

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INTRODUCTION

Variational methods play an important role in mathematics, mechanics and other science and technologies. In the early 16 century, Newton, Leibnitz, Euler, Lagrange and other scientists have studied the variational methods [1]. The origin of the variational methods is the problem of brachistochrone curve, which carries a point-like body from one place to another in the least amount of time. Newton, Leibnitz and Johann Bernoulli solve this problem and their methods give the idea of variational methods. Another problem is the geodesic problem; it is finding the shortest path connecting two points. On a Riemannian manifold M with riemannian metric g, the length of a smooth curve $\gamma : [a,b] \to M$ is defined by the following:

$$L(\gamma)=\sqrt{g_{\gamma}(\dot{\gamma},\dot{\gamma})} \tag{1}$$

The minimizing curves of L in a small enough open set can be obtained by techniques of calculus of variations. The existence and uniqueness problem of geodesics promote the development of geometry. Variational methods have many important applications in differential equations. Solving a differential equation is the same with finding the critical points of some variational functional. This idea was first used in the differential equation of second order. For example we see the next boundary value prblem of linear elliptic equation [2,3]

$$\begin{aligned} -\Delta u &= f(\mathrm{x}), \mathrm{x} \in \Omega \\ u &= 0, x \in \Omega \end{aligned} \tag{2}$$

Here Ω is bounded domain with smooth boundary of the Eucild space. By variational method, one can translate this problem to be finding the critical points of the functional:

$$I(\mathrm{v}) = \int_{\Omega} \frac{1}{2}(|Dv|^2 - fv)dx \tag{3}$$

The process of finding the critical points of this functional is to find the extremal point of tihs functional. Later, peple use the lower semi-continuity condition to study the degenerate elliptic problem:

$$-\nabla\cdot(|\nabla u|^{p-2}\nabla \mathrm{u}) = \mathrm{f}, \mathrm{in}\,\Omega \qquad (4)$$
$$u = 0, x \in \Omega$$

The key point to solve this elliptic problem is finding the extremal point of its corresponding functional. By the usual classic result,we have if M is topological Hausdorff space, and suppose that a functional satisfies the condition of bounded compactness(Heine-Borel property), then E is uniformly bounded from below and attain its infimum on M. However, the condition of bounded compactness is not easy to confirmed, so people try to use some facts that can be checked easily to instead the above classical results.

Suppose V is a reflexive banach space with norm $\|\cdot\|$, and $M \subset V$ is a weakly closed subset of V. Suppose that a functional E is coercive and weak lower semi-continuous on M with respect to V. Then the functional E is bounded from below on M and attains its infimum in M. Using the modified results, one can check that the corresponding functional :

$$E(\mathrm{u}) = \frac{1}{p}\int_{\Omega}|\nabla u|^{p} - \int_{\Omega} fudx \qquad (5)$$

Satisfies the requirment of the modified facts. It is easy to checked that the banach space $W_0^{1,P}(\Omega)$ is reflexive and the the functional is coercive and weak sequentially lower semi-continuous. So the above results implies that there is a minimizer for the functional, and therefore there is a solution for the degenerate elliptic problem.

In general people can not get that a bounded and lower semi-continuous functional get its infimum. Ekeland construct minimizing sequence and show that there exists nearly optimal solutions to some optimization problems. And this principle can be used when the level set is not compact. Compactness becomes more and more important. To overcome this difficulty of lacking compactness, Palis and Smale introduced the concepts of PS-condition: a sequence u_m in V is called a Palais-Smale sequence for E if $|E(\mathrm{u}_m)| \le c$ uniformly in m, while the Differential of the functional $\|DE(\mathrm{u}_m)\| \to 0$ as m $\to$ 0, a functional E is said to satisfy the Palais-Smale condition if any Palais-Smale sequence has a convergent subsequence.

In 1970s, A.Ambrosetti and P.Rabinowitz used the Mountain pass theorem to solve some differential equations[6]. P.Rabinowitz proved the periodic solutions of the Hamiltonian system. Later Benci-Rabinowitz give the linking argument, they can solve more semilinear elliptic boundary problem without symmetry. There are other developments for the variational methods, and the applications in symplectic geometry and dynamical system, we will discuss later.

In this paper, we will first introduce some basic notions of variational methods, show the basic techniques to find the extrema, and give some example to really find the extrema, and last we show the applicaition in symplectic geometry and discuss some future the variational methods maybe ca be used.

1. PREPARATIONS AND LEMMAS

2.1 Preparations

In this section, we briefly introduce the notations of variational methods; give some basic lemmas and properties of Euler equations, Jacobi conditions and some variational problems for geometry, for details we refer to [1, 2, 3, 4, 7].

Definition 1 A function f $\mathbb{R}^n \to \mathbb{R}$ is lower semi continuous at a point $\bar{x}$ if the following identities hold:

$$\liminf_{x\to\bar{x}} f(\mathrm{x}) = f(\bar{x}) \qquad (6)$$

The lower semi continuous is equivalent as the following [13]:

Theorem 2. The following properties of a function f are equivalent

1. F is lower semicontinuous
2. The epigraph set is closed
3. The level set are closed.

Definition 3. Fréchet derivative of a function is defined as following: Let V and W be Banach spaces, and $U \subset V$ be an open subset of V. A function f: U → W is called Fréchet differentiable at $x \in U$ if there exists a bounded linear operator $A_X : \mathrm{V} \to \mathrm{W}$ such that

$$\lim_{h \to 0} \frac{\| f(\mathrm{x}+\mathrm{h}) - \mathrm{f(x)} - \mathrm{A}_x(\mathrm{h}) \|_w}{\|h\|_v} = 0 \tag{7}$$

Now we give the fundamental theorem of calculus of calculus of variations.

Definition 4 If a function $u \in C(\Omega)$ satisfies [1]

$$\int_{\Omega} u(\mathrm{x})\,\mathrm{f(x)} = 0, \forall\, \mathrm{f} \in C_0^{\infty}(\Omega) \tag{8}$$

Then $u = 0$ in Ω .Next we introduce the Euler equation of the variational functional [1]:

Lemma 5 If the functional

$$J[\mathrm{y(x)}] = \int_{x_0}^{x_1} F(\mathrm{x}, \mathrm{y}, \mathrm{y}')dx \tag{9}$$

Satisfies the boundary value conditions

$$y(\mathrm{x}_0) = \mathrm{y}_0, y(\mathrm{x}_1) = \mathrm{y}_1$$

Then the extremal curve should satisfy the following equation

$$F_y - \frac{d}{dx} F_{y'} = 0 \tag{10}$$

Here the function F is of second order. And this equation is also called Euler- Lagrange equation. This equation can also be written as the following

$$F_y - F_{xy'} - F_{yy} y' - F_{y'y} y'' = 0 \tag{11}$$

Once we have the Euler- Lagrange equation, we can translate the extrmal problem to solve differential equations.If there are more variables, the Euler- Lagrange equation can also be found.

Lemma 6 If the functional

$$J[\mathrm{y(x)}, \mathrm{z(x)}] = \int_{x_0}^{x_1} F(\mathrm{x}, \mathrm{y}, \mathrm{y}'\mathrm{z}, \mathrm{z}')dx \tag{12}$$

get its extrema and satisfies the boundary value conditions

$$y(\mathrm{x}_0) = \mathrm{y}_0, y(\mathrm{x}_1) = \mathrm{y}_1, \mathrm{z}(\mathrm{x}_0) = \mathrm{z}_0, \mathrm{z}(\mathrm{x}_1) = z_1 \tag{13}$$

Then there extremal curve satisfies the following Euler- Lagrange equations

$$F_y - \frac{d}{dx} F_{y'} = 0$$
$$F_z - \frac{d}{dx} F_{z'} = 0 \tag{14}$$

If the variational functional has many variables, the Euler- Lagrange equations can also be written in the same way. Moreover, if the functional depend on the higher order differential of the variables,

$$J[\mathrm{y(x)}] = \int_{x_0}^{x_1} F(\mathrm{x, y, y', y''})dx \qquad (15)$$

Suppose that this functional gets its extrema and satisfies the boundary value condition

$$y(\mathrm{x}_0) = \mathrm{y}_0, y(\mathrm{x}_1) = \mathrm{y}_1, y'(\mathrm{x}_0) = \mathrm{y'}_0, y'(\mathrm{x}_1) = \mathrm{y'}_1 \qquad (16)$$

Here the variational functional should be differential of third order, the function y is differential of fourth order, then the Euler- Lagrange equation is as following:

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 0 \qquad (17)$$

This equation is also called Euler-Poisson equation. If there are more variables, the Euler-Poisson equation can also be extended to that case, just note that the functional and the function should have higher differentiation [1].

Next we introduce some sufficient conditions of extrema of Functionals. Consider the functional

$$J[\mathrm{y(x)}] = \int_{x_0}^{x_1} F(\mathrm{x, y, y'})dx \qquad (18)$$

With the boundary value condition

$$y(\mathrm{x}_0) = \mathrm{y}_0, y(\mathrm{x}_1) = \mathrm{y}_1 \qquad (19)$$

If the functional attains its extrema at u(x), then we have the following equation

$$(F_{yy} - \frac{d}{dx}F_{yy'})\mathrm{u} - \frac{d}{dx}(F_{y'y'}\frac{du}{dx}) = 0 \qquad (20)$$

This equation is called the Jacobi equation. It is the same with finding the extrema of functions with one variable; we need some sufficient conditions to determine whether a functional can get its extrema. This is the classical Weiestrass sufficient conditions and Legendre Sufficient conditions. Let

$$E(\mathrm{x, y, y', p}) = F(\mathrm{x, y, y'}) - F(\mathrm{x, y, p}) - (\mathrm{y'} - \mathrm{p})\mathrm{F}_p(\mathrm{x, y, p}) \qquad (21)$$

Theorem 7. If the function y=y(x) is the extremal curve of the functional and satisfies the boundary value condition, and also satisfies the Jacobi conditions, for all near y', the Weiestrass conditions hold,We get its local infmum, if it is less than zero, we get its local Maxima.

2.2 Example

Now we give some examples to show how to determine the extrema.

Example 8. Find the extremal curve of the following functional

$$J(\mathrm{y, z}) = \int_0^{\frac{\pi}{4}} 3z - 9y^2 + y'^2 - 2z'^2\, dx \qquad (22)$$

The boundary conditions are:

$$y(0) = 0, \mathrm{y}(\frac{\pi}{4}) = \sqrt{2}, z(0) = 0, z(\frac{\pi}{4}) = \frac{\pi}{4} \qquad (23)$$

By the Euler- Lagrange equations, we can compute that the function F is

$$F(\mathrm{y, z}) = 3z - 9y^2 + y'^2 - 2z'^2 \qquad (24)$$

So we can compute the $F_y, F_z, F_{y'}, F_{z'}$, so the Euler- Lagrange equations are the following differential equations:

$$18y + 2y'' = 0$$
$$3 + 4z'' = 0 \quad (25)$$

Integral the equations, we can get the solutions of this equations are

$$y = c_1 \cos 3x + c_2 \sin 3x$$
$$z = -\frac{3}{8}x^2 + c_3 x + c_4 \quad (26)$$

According to the boundary conditions, we can get that

$$c_1 = 0, c_4 = 0, c_2 = 2, \mathrm{c}_3 = \frac{32+3\pi}{32} \quad (27)$$

So the extremal curve is

$$y = 2\sin 3x$$
$$z = -\frac{3}{8}x^2 + \frac{32+3\pi}{32}x \quad (28)$$

Next we see the applications of the Jacobi equations, and we just give the idea of the compute and omit the result.

Example 9. Suppose that the functional is the following

$$J(\mathrm{y}) = \int_{x_0}^{x_1} (2\,\mathrm{x}^2 + 3\,\mathrm{y}^2 + 2\,\mathrm{y'}^2)dx \quad (29)$$
$$y(\mathrm{x}_0) = 0, \mathrm{y}(\mathrm{x}_1) = \mathrm{y}_1$$

Find the solution whose Jacobi equation satisfies the initial boundary conditions:

$$u(0) = 0, \mathrm{u}'(0) = 1 \quad (30)$$

First, we can find the function

$$F(\mathrm{x,y,y'}) = 2\,\mathrm{x}^2 + 3\,\mathrm{y}^2 + 2\,\mathrm{y'}^2 \quad (31)$$

And we can compute the Euler equation, then by the definition of Jacobi equation

$$(F_{yy} - \frac{d}{dx}F_{yy'})\,\mathrm{u} - \frac{d}{dx}(F_{y'y'}\frac{du}{dx}) = 0 \quad (32)$$

We can get

$$6u - 4u'' = 0 \quad (33)$$

The solution is

$$u(\mathrm{x}) = \mathrm{d}_1\, e^{\sqrt{1.5x}} + d_2 e^{-\sqrt{1.5x}} \quad (34)$$

And by the initial value of u, we can compute the number d_1, d_2 , we omit the process.

Next we give some other applications of variational methods in symplectic geometry and other dynamical systems. First we introduce the bi-invariant metric on the group of Hamiltonian diffeomorphisms. A manifold is symplectic if it is a smooth manifold and equipped with a closed non-degenerate differential 2-form, the Hamiltonian diffeomorphism is the time-1 map of the following dynamical system

$$i\left(\dot{f}_t\right)\omega = dF_t \tag{35}$$

Hofer first defined a bi-invariant metric on the group of compactly supported symplectic diffeomorphisms of standard symplectic manifold [5], the Hofer norm is defined as following.

$$E(\varphi) = \inf_{f_t}\{\|f_t\| \mid f_1 = \varphi\} \tag{36}$$

Viterbo also defined an invariant distance by generating functions and Polterovich developed this metric on more symplectic manifold [11, 12], finally Lalonde and McDuff extended it to any symplectic manifold [10]. In the definition and proof of Hofer, an Infinite dimensional variational method was used for the variational functional of the Hamiltonian system. An interesting question is whether there exist some more variational results for the geodesic problem under the Hofer metric. How about the variational analysis on the contact bi-invariant metric? Are the similar results for the geodesic problem?

Second we will discuss some applications of variational methods in the biological system. One typical model is the equations for two types of living creatures live in one food source. The modles are the following[8,9]

$$S' = \mathrm{D}(1+\mathrm{be(t)}-\mathrm{S}) - \mathrm{m}_1\frac{x_1 s}{a_1+S} - m_2\frac{x_2 s}{a_2+S} \tag{37}$$

$$x_1' = m_1\frac{x_1 s}{a_1+S} - D_1 x_1 \tag{38}$$

$$x_2' = m_2\frac{x_2 s}{a_2+S} - D_2 x_2 \tag{39}$$

Here x is the number of the living creature, s is the concentration of nutrient solution, mis the rate of growing. Smith use the variational theorems to stduy this problem and analysis the changes of the iving creatures. And this methods can also be used for the analysis of insets in the cereals.

CONCLUSION

In this paper we introduce the history and some techniques in variational methods. We compute some extremas of functional by the Euler equation and the Jacobi equation, and discuss some future applications in symplectic dynamical systems and biological mathematical problems

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