Available online www.jocpr.com

Journal of Chemical and Pharmaceutical Research, 2015, 7(3):2366-2372

Research Article

ISSN : 0975-7384
CODEN(USA) : JCPRC5

Uncertain programming with recourse

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ABSTRACT

In practical decision making process, people often have to deal with indeterminacy data. To handle this situation, many optimization models involved uncertainty are widely discussed. Specifically, in a supply chain, costs of transportation and inventory are important factors in optimizing profits. While, the market demands are often unknown, especially when a new situation arises. In this paper, based on uncertainty theory, a new type of two-stage programming, named uncertain programming with recourse (UPR) is first put forward. Then, by employing the expected value of uncertain variable, an equivalent classic programming of UPR is built. Finally, by regarding the market demands as uncertain variables, UPR model is used to solve the integrating transportation and inventory problem under uncertainty.

Key words: uncertain programming; uncertainty theory; two-stage programming

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INTRODUCTION

Two-stage stochastic programming was first put forward by Dantzig [6] in 1955. The linear programming he discussed was divided into two or more stages. When the first stage was determined, later stages depended on the earlier stages and the random demands. By minimizing the expected value of the objective function, an equivalent linear programming model was obtained.

After that, numerous research works have been undertaken to explore its properties and numerical algorithms. In 1967, Agizy [11] discussed two-stage programming with discrete distribution function. Walkup and Wets [10] generalized Danzig's model and considered the random coefficients in the second stage. Shor and Shchepakin [13] gave a penalty vector algorithm to solve the two-stage model in 1968. Kall and Peter [14] proposed an approximating method for solving two-stage stochastic programming with random variables obeying discrete distributions in 1979. Afterwards, many researchers developed the two-stage stochastic programming (Louveaux [1], Birge [2], Lustig [3], Ruszczynski and Swietanowski [4], Dai [12]).

Fuzzy set theory, introduced by Zadeh [7] [8], has been widely applied to handling fuzziness. In 2005, based on credibility theory, Liu [9] introduced a two-stage fuzzy programming. And then Liu [5] studied two-stage fuzzy random programming in 2007. While, the fuzzy set theory is facing challenges by many researchers, who hold different opinions that human uncertainty can't be manifested as fuzziness. Liu [21] pointed out that the measure of union of events is not necessarily the maximum of measures of individual events.

As everyone knows, we can employ probability when the estimated probability distribution is approximately equal to real frequency. While, in practical problems, we may encounter the situation that it is difficult to obtain observed data. When this happens, people have to depend on experts' opinions to evaluate the belief degree that each event will occur. However, Kahneman and Tversky [24] pointed out that human tends to overweight events which they are unsure of. In this case, some counterintuitive things will arise if we insist on using probability (Liu [21]). In order to handle this situation, an uncertainty theory was founded by Liu [17] in 2007 and refined by Liu [20] in 2010. After that, many researchers widely studied the uncertainty theory and made significative progress.

In 2009, uncertain programming was first proposed by Liu [18]. Then, an uncertain multi-objective programming and an uncertain goal programming were provided by Liu and Chen [22]. Later in the time, an uncertain multi-level programming was put forward by Liu and Yao [23]. At present, uncertain programming has wined widespread use in engineering, management and design, such as inventory problem [25], Chinese postman problem [26], and project scheduling problem [27].

The context of the above factors results in the motivation of this study. In this paper, uncertain programming with recourse is first proposed. We shall first briefly introduce uncertainty theory and related concepts. In the next section, after a formulation of the uncertain programming with recourse, an equivalent classical programming model is given. Then, the effectiveness of this method is verified by an example. Finally, conclusions and future work are summarized.

PRELIMINARY

In this section we will provide a brief introduction of uncertain measure, uncertain variable, expected value and uncertain programming, which will be used throughout this paper.

Definition 1. (Liu [17]) Let Γ be a nonempty set and L be a σ-algebra on Γ. Each element in L is called an event. A set function M from L to $[0,1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1. (Normality Axiom) $\mathrm{M}\{\Gamma\}=1$ for the universal set Γ.

Axiom 2. (Duality Axiom) $\mathrm{M}\{\Lambda\}+\mathrm{M}\{\Lambda^c\}=1$ for any event Λ.

Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1,\Lambda_2,\cdots$, we have

$$\mathrm{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\}\le\sum_{i=1}^{\infty}\mathrm{M}\{\Lambda_i\}.$$

The triplet $(\Gamma,\mathrm{L},\mathrm{M})$ is called an uncertainty space.

In 2010, Liu [20] defined product uncertain measure via the fourth axiom of uncertainty theory.

Axiom 4. (Product Axiom) Let $(\Gamma_k,\mathrm{L}_k,\mathrm{M}_k)$ be uncertainty spaces for $k=1,2,\cdots$ Then the product uncertain measure M is an uncertain measure satisfying

$$\mathrm{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\}=\bigwedge_{k=1}^{\infty}\mathrm{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from L for $k=1,2,\cdots$, respectively.

An uncertain variable is a real valued function on an uncertainty space, which is defined as follows.

Definition 2. (Liu [17]) Let $(\Gamma,\mathrm{L},\mathrm{M})$ be an uncertainty space. An uncertain variable is a measurable function from an uncertainty space Γ to the set of real numbers, i.e., for any Borel set B of real numbers, the set $\xi^{-1}(B)=\{\gamma\in\Gamma\mid\xi(\gamma)\in B\}$ is an event.

In order to describe uncertain variables, Liu [17] introduced uncertainty distribution.

Definition 3. (Liu [17]) The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathrm{M}\{\xi \le x\}$$

for any real number x.

Definition 4. (Liu [17]) An uncertain variable ξ is called linear if it has a linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if} \quad x < a \\ (x-a)/(b-a), & \text{if} \quad a \le x < b \\ 1, & \text{if} \quad x \ge b \end{cases}$$

denoted by $L(a,b)$ where a and b are real numbers with $a < b$.

Definition 5. (Liu [19]) The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if

$$\mathrm{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathrm{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \cdots, B_n$ of real numbers.

Theorem 1. (Liu [17]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain variables and f be a real-valued measurable function. Then $f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable.

To represent the average value of an uncertain variable in the sense of uncertain measure, the expected value is defined as follows.

Definition 6. (Liu [17]) Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathrm{M}\{\xi \ge r\}dr - \int_{-\infty}^{0} \mathrm{M}\{\xi \le r\}dr \tag{1}$$

provided that at least one of the two integrals is finite.

Definition 7. (Liu [17]) Let ξ be an uncertain variable with uncertainty distribution Φ. If the expected value exists, then

$$E[\xi] = \int_0^{+\infty} (1-\Phi(x))dx - \int_{-\infty}^{0} \Phi(x)dx. \tag{2}$$

For calculating the expected value by inverse uncertainty distribution, Liu and Ha [15] proved the following theorem.

Theorem 2. (Liu and Ha [15]) Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then the uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an expected value

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))d\alpha. \tag{3}$$

Theorem 3. (Liu [20]) For independent uncertain variables ξ and η, we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]$$

for any real numbers a and b.

Uncertain programming is a type of mathematical programming involving uncertain variables which can be formulated as follows.

Let x be a decision vector, and ξ be an uncertain vector. Assume that f is an objective function, and $g_j(x,\xi),\ j=1,2,\cdots,p$ are constraint functions. Note that the uncertain constraints $g_j(x,\xi),\ j=1,2,\cdot s,p$ do not define a classic feasible set. We suppose that $\alpha_1,\alpha_2,\cdots,\alpha_p$ are confidence levels of the uncertain constraints, which represent the uncertain constraints hold with the specified confidence levels. Then the uncertain programming (Liu [18]) can be built as follows,

$$\begin{cases} \min\limits_{x} E[f(x,\xi)] \\ \text{subject to}: \\ \mathrm{M}\{g_j(x,\xi) \le 0\} \ge \alpha_j,\ j=1,\cdots,p. \end{cases} \tag{4}$$

UNCERTAIN PROGRAMMING WITH RECOURSE

In literature, many researchers have discussed the recourse problem of stochastic and fuzzy programming which have been applied to many decision problems in indeterminacy environments. As mentioned previously, in many real world situations, we are provided incomplete information because of time pressure and lack of data. This consideration inspires us to consider the recourse problem of uncertain programming. Aiming at taking uncertainty theory as the theoretical basis, we propose a new class of recourse problem, named uncertain programming with recourse (UPR), in this section. By using uncertainty theory listed in the preliminary, UPR is converted into classic programming. Besides, we consider the case that the objective function in the recourse stage is nonlinear. Last, we give an example to illustrate the UPR model.

Uncertain programming with recourse is formulated as follows,

$$\begin{cases} \min\limits_{x} Z(x) = c^{\mathrm{T}}x + E\left[\min\limits_{y} q^{\mathrm{T}} y\right] \\ \text{subject to}: \\ Ax \ge b \\ Wx + y \ge \xi \\ x \ge 0, y \ge 0 \end{cases} \tag{5}$$

where $x \in R^n$ is a decision vector of the first stage problem, $y \in R^m$ is a decision vector of the recourse stage and $c \in R^n, q \in R^m, b \in R^n, A \in R^{Mn}, W \in R^{mn}$, in which M, n, m are positive integers and q is a positive vector. The uncertain vector ξ is denoted by $(\xi_1,\xi_2,\cdots,\xi_m)^{\mathrm{T}}$, in which $\xi_1,\xi_2,\cdots,\xi_m$ are independent uncertain variables with uncertainty distributions $\Phi_1,\Phi_2,\cdots,\Phi_m$. In this paper, 0 represents the suitable zero matrix.

Suppose there exists a feasible solution of the first stage problem, and the feasible region is denoted by D_1,

$$D_1 = \{x \in R^n \mid A(x) \ge b, x \ge 0\}. \tag{6}$$

The recourse stage problem is referred to

$$\begin{cases} E\left[\min\limits_{y} q^{\mathrm{T}} y\right] \\ \text{subject to:} \\ \quad Wx + y \geq \xi \\ \quad y \geq 0. \end{cases} \tag{7}$$

We suppose that, for all x in D_1, the recourse stage problem has at least one feasible solution. Let

$$q(x,\xi) = \min_{y}\{q^{\mathrm{T}} y \mid Wx + y \geq \xi, y \geq 0\} = q^{\mathrm{T}} y^{\mathring{a}}. \tag{8}$$

Since $q_1, q_2, \ldots, q_m > 0$, then we have $y^{\mathring{a}} = ((\xi_1 - w_1 x)^+, (\xi_2 - w_2 x)^+, \cdots, (\xi_m - w_m x)^+)$, in which w_i represent the i th lines of W and $(\xi_i - w_i x)^+$ represent the positive parts of $(\xi_i - w_i x)$ for all $i = 1, 2, \cdots, m$. Thus,

$$q(x,\xi) = \sum_{i=1}^{m} q_i (\xi_i - w_i x)^+. \tag{9}$$

Suppose that the recourse stage problem has finite optimal solution, which means

$$D_2 = \{x \in R^{N_1} \mid Q(x,\xi) < +\infty\}. \tag{10}$$

Obviously, $q(x,\xi)$ is an uncertain variable and its expected value is denoted by $Q(x)$,

$$Q(x) = E[q(x,\xi)] = E\left[\sum_{i=1}^{m} q_i (\xi_i - w_i x)^+\right]. \tag{11}$$

Since $q(x,\xi)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$, and by Theorem 2.3.1 we obtain

$$Q(x) = E\left[q(x,\xi)\right] = \int_0^1 \sum_{i=1}^{m} q_i \left(\Phi_i^{-1}(\alpha) - w_i x\right)^+ d\alpha. \tag{12}$$

The expected value $Q(x)$ is called the recourse function of the recourse stage programming. Thereupon we obtain a deterministic equivalent programming of the UPR problem

$$\begin{cases} \min\limits_{x} c^{\mathrm{T}} x + Q(x) \\ \quad \text{subject to:} \\ \quad\quad x \in D \end{cases} \tag{13}$$

in which $D = D_1 \bigcap D_2$. As a result, the UPR is converted into a classic programming under some assumptions.

In this section, we will discuss a more general form of UPR with nonlinear objective function $q(x,y)$ in the recourse stage,

$$\begin{cases} \min\limits_{x} Z(x) = c^{\mathrm{T}} x + E\left[\min\limits_{y} q(x,y)\right] \\ \quad \text{subject to:} \\ \quad\quad Ax \geq b \\ \quad\quad Wx + y \geq \xi \\ \quad\quad x \geq 0, y \geq 0. \end{cases} \tag{14}$$

Suppose $q(x,y)=q(x,y_1,y_2,\cdots,y_m)$ is strictly increasing with respect to $y_1,y_2,\cdots,y_m$. The rest notations are identical with Equation (5). By solving the recourse problem, we obtain

$$q(x,\xi)=\min_y\{q(x,y)\,|\,Wx+y\geq\xi, y\geq 0\}$$
$$=q(x,(\xi-Wx)^+).$$

Since $q(x,y)$ is strictly increasing with respect to $\xi_1,\xi_2,\cdots,\xi_m$, then, by Theorem 2.3.1, the expected value of the uncertain variable $q(x,y)$ in the recourse stage can be calculated by the following formula,

$$Q(x)=E[q(x,y)]=\int_0^1 q\left(x,(\Phi_1^{-1}(\alpha)-w_1x)^+,(\Phi_2^{-1}(\alpha)-w_2x)^+,\cdots,(\Phi_r^{-1}(\alpha)-w_rx)^+\right)d\alpha. \quad (15)$$

Thus, through a similar method with the previous section, we can convert the UPR problem (14) to equivalent crisp programming problem.

we give an example to illustrate the above method.

Example 1. Giving a UPR problem, suppose $x\in R$ is a decision variable of the first stage and $y\in R$ is a decision variable of the recourse stage. The uncertain variable ξ obeys a linear uncertainty distribution $\Phi(x)=L[1,3]$,

$$\begin{cases}\min\limits_x x+E[\min\limits_y y^2] \\ \quad\text{subject to:} \\ \quad\quad x-2\geq 0 \\ \quad -x-\xi+y\geq 0 \\ \quad\quad x,y\geq 0.\end{cases} \quad (16)$$

By Definition 2.3.3, the uncertainty distribution of ξ is

$$\Phi(x)=\begin{cases}0, & \text{if} \quad x<1 \\ (x-1)/2, & \text{if} \quad 1\leq x<3 \\ 1, & \text{if} \quad x\geq 3.\end{cases}$$

The recourse function is

$$q(x,\xi)=\min_y\{y^2\,|\,-x-\xi+y\geq 0, y\geq 0\}=(x+\xi)^2.$$

Obviously, $q(x,\xi)$ is strictly increasing with respect to ξ. By Equation (14), we have

$$Q(x)=E[q(x,\xi)]=E[(x+\xi)^2]$$
$$=\int_0^1(x+\Phi^{-1}(\alpha))^2d\alpha=\frac{13}{3}+4x+x^2.$$

Then we obtain the equivalent classic programming

$$\begin{cases}\min\limits_x x^2+5x+\frac{13}{3} \\ \quad\text{subject to:} \\ \quad\quad x-2\geq 0 \\ \quad\quad x\geq 0.\end{cases} \quad (17)$$

Thus, the optimal value is $z = 55/3$, when $x = 2$.

CONCLUSION

In this paper, we firstly proposed a general form of uncertain programming with recourse and investigated the equivalent classic programming. Finally, a numerical example was put forward to illustrate the effectiveness of this method.

Acknowledgments
This work was supported by the Foundation of Anhui Educational Committee (KJ2013B165).

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