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The distributions and moments of the first entrance time for nonhomogeneous (H,Q) process and their application in nonhomogeneous semi-markov process

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ABSTRACT

In this paper, we study one kind of Markov Skeleton processes, nonhomogeneous (H,Q) processes, and we mainly focus on their distributions and moments of the first entrance time. With detailed analysis, we obtain their recursive formula and some related properties. And as one kind of nonhomogeneous (H;Q) processes, Nonhomogeneous Semi-Markov Process, we study it's distributions and moments of the first entrance time, and we also obtain their recursive formula.

Keywords: Nonhomogeneous (H,Q) process, distributions and moments, first entrance time, Nonhomogeneous Semi-Markov Process

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INTRODUCTION

In applied probability fields, we usually find that there are a large class of stochastic processes which are not Markov Processes(MP), but there exist a sequence of stopping times (τ_n), at which the stochastic processes have the Markov property. Stimulated by this discovery , Hou et. al [1,2] named them Markov skeleton processes (MPS). Usually, MP can be regarded as a special case of MSP.

One of the result in Hou et.[2] is about the minimal nonnegative solution of the "backward equation" which is determined by two variables called binary characteristic (H,Q), and through (H,Q), the nonnegative solution can expressed by an explicit formula. We also call such kind of process as (H,Q) process. And homogeneous (H,Q) process and some application have been studied in [2]-[7] that provide some preliminary theoretic foundation, while as the process being nonhomogeneous, it will become more generous. In this paper, we will discuss the distributions and moments of the first entrance time for the nonhomogeneous (H,Q) pocess and their application in nonhomogeneous semi-Markov process
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The remainder of this paper is organized as follows. Section 2 gives the definition of nonhomgeneous (H,Q) process. Section 3 discusses their distributions and moments of the first entrance time and some related properties. In section 4, we discuss their application in nonhomogeneous semi-Markov process

NONHOMOGENEOUS (H,Q) PROCESS

Let (Ω, F, P) be a complete probability space, $X = \{X(t,\omega), 0 \le t < \tau(\omega)\}$ is a stochastic process defined on (Ω, F, P) with values in (E, ε), where (E, ε) be a measurable space.

Definition 1 A stochastic process $X = \{X(t,\omega), 0 \le t < \tau(\omega)\}$ is called a nonhomogeneous (H,Q)- process, if there exists a sequence of stopping times $\{\tau_n\}_{n\ge 0}$ which satisfying the following properties,

(i) $0 = \tau_0 \le \tau_1 \le \tau_2 \le \cdots, \tau = \lim_{n\to\infty} \tau_n, P - a.e$;

(ii) $E[X(\tau_n + t) \in A, \tau_{n+1} - \tau_n > t \mid X(\tau_n), \tau_n, X(\tau_{n-1}), \tau_{n-1}, \cdots, X(0)]$

$= E[X(\tau_n + t) \in A, \tau_{n+1} - \tau_n > t \mid X(\tau_n)] = h^{(n+1)}(t, X(\tau_n), A), \qquad n \ge 0, t \ge 0, A \in \varepsilon;$

(iii) $E[X(\tau_{n+1}) \in A, \tau_{n+1} - \tau_n \le t \mid X(\tau_n), \tau_n, X(\tau_{n-1}), \tau_{n-1}, \cdots, X(0)]$

$= E[X(\tau_{n+1}) \in A, \tau_{n+1} - \tau_n \le t \mid X(\tau_n)] = q^{(n+1)}(t, X(\tau_n), A), \qquad n \ge 0, t \ge 0, A \in \varepsilon.$

For fixed A, $h^{(n)}(t,x,A)$ and $q^{(n)}(t,x,A)$ are measued functions about the variables t and x; and for fixed t and x, $h^{(n)}(t,x,A)$ and $q^{(n)}(t,x,A)$ are quasi-distributions on (E, ε).

Let $q^{(n)}(x,A) = \lim_{t\to\infty} q^{(n)}(t,x,A)$, then by (iii), we know that $\{X(\tau_n)\}_{(n\ge 0)}$ is nonhomogeneous Markov process with staionary transition probabilities $\{q^{(n)}(x,A), x \in E, A \in \varepsilon\}$.

Intuitively, we decompose the process X into denumerable parts $\{X(t,\omega), \tau_{n-1} \le t < \tau_n\}_{n\ge 1}$ by a series of increasing Markov times $\{\tau_n\}_{n\ge 0}$, and nonhomogeneous (H,Q) process is determined by $(h^{(n)}(t,x,A))$ and $(q^{(n)}(t,x,A))$. Obviously, nonhomogeneous (H,Q) process include a kind of process with more conditions.

Suppose A, H are closed subsets of E, and $A \ne \Phi$. In what follows, we define some series of function with indexed as follows.

$$\sigma_A(\omega) = \begin{cases} \inf(t, 0 < t < \tau(\omega), X_t(\omega) \in A, & if \quad 0 < t < \tau(\omega), X_t(\omega) \ne \Phi \\ \quad +\infty, & otherwise. \end{cases}$$

$$\sigma_A(\omega) = \begin{cases} \sigma_A(\omega), \ if \quad \sigma_A(\omega) \le \sigma_H(\omega) \\ \quad +\infty, \qquad otherwise. \end{cases}$$

For $n \in N, \lambda \ge 0$ and $p = 1,2,\cdots$, letting

$${}_H f_{xA}^{(n)}(t) = P({}_H\sigma_A \le t, \tau_{n-1} <_H \sigma_A \le \tau_n \mid X(0) = x), \quad {}_H f_{xA}(t) = P({}_H\sigma_A \le t \mid X(0) = x),$$

$${}_H\varphi_{xA}^{(n)}(\lambda) = \int_0^\infty e^{-\lambda t} d_H f_{xA}^{(n)}(t), \qquad {}_H\varphi_{xA}(\lambda) = \int_0^\infty e^{-\lambda t} d_H f_{xA}(t),$$

$${}_H m_{xA}^{n,p}(\lambda) = \int_0^\infty t^p d_H f_{xA}^{(n)}(t), \qquad {}_H m_{xA}^{(p)}(\lambda) = \int_0^\infty t^p d_H f_{xA}(t),$$

and ${}_H f_{xA}^* = {}_H\varphi_{xA}(0) = {}_H m_{xA}^{(0)} = P({}_H\sigma_A(\omega) < \infty \mid X(0) = x).$

In what follows, we suppose for $n > 0, t \ge 0, B \in \varepsilon$, nonhomogeneous (H,Q) process X satisfies the following equations:

$$E(X(\tau_n) \in B \mid N_{\tau_n^-}] = E[X(\tau_n) \in B \mid X(\tau_n^-)],$$

$$E(X(\tau_n) \in B, \tau_n \le t \mid N_{\tau_n^-}] = E[X(\tau_n) \in B \mid N_{\tau_n^-}] \cdot E[\tau_n \le t \mid N_{\tau_n^-}],$$

where, $N_{\tau_n^-} = \sigma\{B \cap \{t < \tau_n\}; B \in \sigma\{X(s); s \ge t\}, t \ge 0\}$.

DISTRIBUTIONS AND MONMENTS OF THE FIRST TIME

In the following, we will discuss the distributions and moments of the first time for nonhomogeneous (H,Q) process.

Theorem 1 For nonhomogeneous (H,Q) process, ${}_H f_{iA}(t)$, ${}_H \varphi_{iA}(\lambda)$ and ${}_H m_{iA}^{p}(t)$ satisfy the following equations respectively,

$${}_H f_{iA}(t)=\sum_{n=1}^{\infty}({}_H f_{iA}^{(n)}(t)),\quad {}_H\varphi_{xA}(\lambda)=\sum_{n=1}^{\infty}({}_H\varphi_{xA}^{(n)}(\lambda)),\quad {}_H m_{xA}^{(p)}=\sum_{n=1}^{\infty}({}_H m_{xA}^{n,p}).$$

Proof: Their definition implies that the three questions above are true.

We now introduce the following Lemma (the proof referring to [6],page 97).

Lemma 1 $\forall n\geq 1$, and $B\in\varepsilon$, then

$$E(X(\tau_n)\in B|\ N_{\tau_n^-}\vee\sigma(\tau_n)]=E[X(\tau_n)\in B|\ N_{\tau_n^-}]$$

Where $N_{\tau_n^-}\vee\sigma(\tau_n)$ is the smallest σ region including $N_{\tau_n^-}$ and $\sigma(\tau_n)$.

For $n=1,2,\cdots$, letting

$$\pi_k(x,B)=P(X(\tau_k)\in B|\ X(\tau_k^-)=x),k=1,2,\cdots;$$

$$\varphi_k(x,B)=P({}_H\sigma_A\geq\tau_{k+1},X(\tau_{k+1}^-)\in B|\ X(\tau_k)=x),k=0,1,2,\cdots;$$

$$\varphi\ (x,B)=\varphi_0(x,B)=P({}_H\sigma_A\geq\tau_1,X(\tau_1^-)\in B|\ X(0)=x),;$$

$$\phi(x,B)=P({}_H\sigma_A\leq t,0<_H\sigma_A<\tau_1,|\ X(0)=x);$$

$${}_H f_{xA}^{(k,n)}(t)=P({}_H\sigma_A\leq t,\tau_{n-1}<_H\sigma_A\leq\tau_n|\ X(\tau_k)=x),k=1,2,\cdots n-1;$$

$${}_H\varphi_{xA}^{(k,n)}(\lambda)=\int_0^{\infty}e^{-\lambda t}d_H f_{xA}^{(k,n)}(t);\qquad {}_H m_{xA}^{(k,n),p}=\int_0^{\infty}t^p d_H f_{xA}^{(k,n)}(t),p=1,2,\cdots.$$

Lemma 2 For any $B\subset E\setminus(A\cup H)$ and $k=0,1,2,\cdots$, we have the following equations

$$P(X(\tau_{k+1})\in B,{}_H\sigma_A\geq\tau_{k+1}|\ X(\tau_k)=x]=\int\pi_{k+1}(z,B)\cdot\varphi_k(x,dz)$$

Proof: $P(X(\tau_{k+1})\in B,{}_H\sigma_A\geq\tau_{k+1}|X(\tau_k)=x)$

$$=E[{}_H\sigma_A\geq\tau_{k+1},E[X(\tau_{k+1})\in B|\ N_{\tau_{\setminus k+1}^-}\vee\sigma(\tau_{k+1})]|X(\tau_k)=x]$$

$$=E[{}_H\sigma_A\geq\tau_{k+1},E[X(\tau_{k+1})\in B|\ X(\tau_{k+1}^-)]|X(\tau_k)=x]$$

$$=\int P(X(\tau_{k+!})\in B|\ X(\tau_{k+1}^-)=z)\cdot P({}_H\sigma_A\geq\tau_{k+1},X(\tau_{k+1}^-)\in dz|\ X(\tau_k)=x)$$

$$=\int\pi_{k+1}(z,B)\cdot\varphi_k(x,dz).$$

Lemma 3 For fixed $n\in N$, we have the recursive formula as the following

$${}_H f_{x,A}^{(k,n+1)}(t)=\int\varphi_k(x,dz)\int_{E\setminus(A\cup H)}({}_H f_{yA}^{(k+1,n+1)}(t))\cdot\pi_{k+1}(z,dy),$$

And then for $k=1,2,\cdots,n-1$, we have

$${}_H\varphi_{x,A}^{(k,n+1)}(t)=\int\varphi_k(x,dz)\int_{E\setminus(A\cup H)}({}_H\varphi_{yA}^{(k+1,n+1)}(t))\cdot\pi_{k+1}(z,dy),$$

$${}_H m_{x,A}^{(k,n+1),p}(t)=\int\varphi_k(x,dz)\int_{E\setminus(A\cup H)}({}_H m_{yA}^{(k+1,n+1),p}(t))\cdot\pi_{k+1}(z,dy),$$

Proof: ${}_H f_{x,A}^{(k,n+1)}(t)=P({}_H\sigma_A\leq t,\tau_n<_H\sigma_A\leq\tau_{n+1}|X(\tau_k)=x)$

$$=P(X(\tau_{k+1})\in E\setminus(A\cup H),{}_H\sigma_A\leq t,\tau_n<_H\sigma_A\leq\tau_{n+1}|X(\tau_k)=x)$$

$$=\int_{\Omega}E[{}_H\sigma_A\leq t,\tau_n<_H\sigma_A\leq\tau_{n+1}|X(\tau_{k+1}),\tau_{k+1},X(\tau_k)]\cdot I_{\{X(\tau_{k+1})\in E\setminus(A\cup H)\}}\cdot I_{\{{}_H\tau_A\geq\tau_{k+1}\}}P(d\omega|X(\tau_k)=x]$$

$$=\int_{E\setminus(A\cup H)}P({}_H\sigma_A\leq t,\tau_n<_H\sigma_A\leq\tau_{n+1}|X(\tau_{k+1})=y]\cdot P(X(\tau_{k+1})\in dy,{}_H\sigma_A\geq\tau_{k+1}|X(\tau_k)=x]$$

$$=\int_{E\backslash(A\cup H)}({}_Hf_{yA}^{(k+1,n+1)}(t))\cdot P(X(\tau_{k+1})\in dy,{}_H\sigma_A\geq\tau_{k+1}\big|X(\tau_k)=x]$$
$$=\int\varphi_k(x,dz)\int_{E\backslash(A\cup H)}({}_Hf_{yA}^{(k+1,n+1)}(t))\cdot\pi_{k+1}(z,dy).$$

We have equation (4) immediately by taking **Laplace-Stieltjes** transformation about equation (3), and then we get the equation (5) considering the definition of ${}_H m_{iA}^{(k,n+1),p}$.

Letting $\varphi^*(t,x,B)=P({}_H\sigma_A\geq\tau_1,\tau\leq t,X(\tau_1^-)\in B\big|\ X(0)=x)$, we have the theorem as follows,

Theorem 2 $\{{}_H f_{iA}^{(n)}(t)\}$ are determined by the following recursive formula,

$${}_H f_{x,A}^{(1)}(t)=\phi(t,x)+\int_{E\backslash(A\cup H)}\pi_1(z,A)\cdot\varphi^*(t,x,dz)$$
$${}_H f_{x,A}^{(n+1)}(t)=\int\varphi(x,dz)\int_{E\backslash(A\cup H)}({}_Hf_{yA}^{(1,n+1)}(t))\cdot\pi_1(z,dy).$$

Proof: ${}_H f_{xA}^{(1)}(t)=P({}_H\sigma_A\leq t,0<{}_H\sigma_A\leq\tau_1\big|X(0)=x]$

$$=P({}_H\sigma_A\leq t,0<{}_H\sigma_A<\tau_1\big|X(0)=x]+P({}_H\sigma_A\leq t,{}_H\sigma_A=\tau_1\big|X(0)=x]$$
$$=\phi(t,x)+P({}_H\sigma_A\leq t,{}_H\sigma_A=\tau_1\big|X(0)=x].$$

While $P({}_H\sigma_A\leq t,{}_H\sigma_A=\tau_1\big|X(0)=x]=P(X(\tau_1)\in A,{}_H\sigma_A\geq\tau_1,\tau_1\leq t\big|X(0)=x]$

$$=E[E[X(\tau_1)\in A,{}_H\sigma_A\geq\tau_1,\tau_1\leq t\big|\ N_{\tau_1^-}\vee\sigma(\tau_1)]\big|X(0)=x]$$
$$=E[{}_H\sigma_A\geq\tau_1,\tau_1\leq t,E[X(\tau_1)\in A\big|\ N_{\tau_1^-}\vee\sigma(\tau_1)]\big|X(0)=x]$$
$$=E[{}_H\sigma_A\geq\tau_1,\tau_1\leq t,E[X(\tau_1)\in A\big|\ X(\tau_1^-)]\big|X(0)=x]=\int_{E\backslash(A\cup H)}\pi_1(z,A)\cdot\varphi^*(t,x,dz).$$

Then we have ${}_H f_{x,A}^{(1)}(t)=\phi(t,x)+\int_{E\backslash(A\cup H)}\pi_1(z,A)\cdot\varphi^*(t,x,dz)$, and then we get the recursive formula about ${}_H f_{x,A}^{(n)}(t),n\geq 1$ immediately by letting $k=0$ in the proof of Lemma 3.

Theorem 3 $\{{}_H\varphi_{xA}^{(n)}(\lambda)\}$ are determined by the following recursive formula,

$${}_H\varphi_{xA}^{(1)}(\lambda)=\psi(\lambda,x)+\int_{E\backslash(A\cup H)}\pi_1(z,A)\cdot\Phi(\lambda,x,dz)$$
$${}_H\varphi_{xA}^{(n+1)}(\lambda)=\int\varphi(x,dz)\int_{E\backslash(A\cup H)}({}_H\varphi_{yA}^{(1,n+1)}(t))\cdot\pi_1(z,dy).$$

Where, $\psi(\lambda,x)=\int_0^\infty e^{-\lambda t}d\phi(t,x),$ and $\Phi(\lambda,x,A)=\int_0^\infty e^{-\lambda t}d\varphi^*(t,x,A),$

Proof : We get the conclusion directly by taking **Laplace-Stieltjes** transformation for ${}_H\varphi_{xA}^{(n)}(\lambda)$.

Theorem 4 $\{{}_H m_{xA}^{n,p}\}$ are determined by the following recursive formula,

$${}_H m_{xA}^{1,p}=\int_0^\infty t^p\phi(dt,x)+\int_0^\infty t^p\int_{E\backslash(A\cup H)}\pi_1(z,A)\cdot\varphi^*(dt,x,dz)$$
$${}_H m_{xA}^{n+1,p}=\int\varphi(x,dz)\int_{E\backslash(A\cup H)}({}_H m_{yA}^{(1,n+1),p})\cdot\pi_1(z,dy).$$

Proof : The definition of ${}_H m_{xA}^{n,p}$ and Theorem 1 imply that the conclusions in above Theorem is true.

APPLICATION IN NONHOMOGENEOUS SEMI-MARKOV PROCESS

Definition 2 The matrix $Q(t)=(Q_{ij}(t),i,j\in E)$ is called Semi-Markov matrix, if for any $i,j\in E,Q_{ij}(t)$ has the following properties.

(iv) $Q_{ij}(t)=0,t<0$ and $\sum_{j\in E} Q_{ij}(t)\le 1,-\infty<t<+\infty$

(v) $Q_{ij}(t)$ being a nondecreasing and right continuous function, $(-\infty<t<+\infty)$;

Definition 3 The stochastic process $X=\{X(t,\omega),0\le t<\tau(\omega)\}$ is called a nonhomogeneous Semi-Markov process, if there are series of stopping time $\{\tau_n\}_{n\ge 0}$ which satisfy,

(vi) $0=\tau_0\le\tau_1\le\tau_2\le\cdots,\tau=\lim_{n\to\infty}\tau_n,P-a.e$;

(vii) $X(t,\cdot)=X(\tau_n,\cdot),\tau_n\le t<\tau_{n+1},P-a.e$;

(viii) $\{X_n=X(\tau_n,\cdot),n=0,1,2\cdots\}$ being a nonhomogeneous Markov chains, i.e. $P_{ij}^{(n)}=P(X(\tau_n)=j|X(\tau_{n-1})=i),n=1,2,\cdots$

Proposition 1 Assume that $X=\{X(t,\omega),0\le t<\tau(\omega)\}$ is a nonhomogeneous Semi-Markov process, and for $n=1,2,\cdots$, letting $Q_{ij}^{(n)}=P(X(\tau_n)=j,\tau_n-\tau_{n-1}\le t|X(\tau_{n-1})=i),$ then $Q_{ij}^{(n)}(t)$ is Semi-Markov matrix.

Proof: We omit its proof for it being trivial.

Proposition 2 If $X=\{X(t,\omega),0\le t<\tau(\omega)\}$ is a nonhomogeneous Semi-Markov process, then about the series of the stopping times $\{\tau_n\}_{n\ge 0}$ and $(h^n(t,i,j))$ and $(q^n(t,i,j))$, X is a nonhomogeneous (H,Q)-Process.

Proof: $P(X(\tau_{n-1}+t)=j,\tau_n-\tau_{n-1}>t|X(\tau_{n-1})=i)$

$=P(X(\tau_{n-1}+t)=j, there\ \ is no\ \ jump\ \ in\ \ [\tau_{n-1},\tau_{n-1}+t]|X(\tau_{n-1})=i)$

$=\delta_{ij}[1-P(X\ \ leaves\ \ off\ \ i\ \ in\ \ [\tau_{n-1},\tau_{n-1}+t]|X(\tau_{n-1})=i]$

$=\delta_{ij}[1-P(\tau_n-\tau_{n-1}\le t|X(\tau_{n-1})=i)]=\delta_{ij}[1-\sum_{k\in E} Q_{ik}^{(n)}(t)]=h^n(t,i,j)$.

For the convenience, we suppose that E is a numberable set, and $X=\{X(t),0\le t<\tau\}$ is nonhomogeneous (H,Q)-process about the stopping time $\{\tau_n\}_{n\ge 0}$.

Lemma 4 If $X=\{X(t),0\le t<\tau\}$ is a nonhomogeneous Semi-Markov process, then we have the equations as the following,

${}_H f_{iA}^{(n)}(t)=P(X({}_H\sigma_A\le t,{}_H\sigma_A=\tau_n(\omega)|X(0)=i)$;

$\pi_k(i,B)=\sum_{k\in B} P_{ik}^{(k)},k=1,2,\cdots;\quad \phi(t,i)=0,\quad \varphi(i,B)=I_{\{i\in B\}}$.

Proof: We have the result above immediately base on their definition.

Theorem 5 If $X=\{X(t),0\le t<\tau\}$ is a nonhomogeneous Semi-Markov process, then for $p=1,2,\cdots$, we have the following equations,

${}_H f_{iA}(t)=\sum_{n=1}^{\infty}({}_H f_{iA}^{(n)}(t))$, ${}_H\varphi_{iA}(\lambda)=\sum_{n=1}^{\infty}({}_H\varphi_{iA}^{(n)}(\lambda))$, ${}_H m_{iA}^{(p)}=\sum_{n=1}^{\infty}({}_H m_{iA}^{n,p})$.

Proof: The proof is trivial by considering their definition.

Lemma 5 If $X=\{X(t),0\le t<\tau\}$ is a nonhomogeneous Semi-Markov process, then for fixed $n(n\ge 1)$, sequence $\{{}_H f_{iA}^{(k,n+1)}(t)\}$ satisfy the following recursive formula,

$_H f_{iA}^{(k,n+1)}(t)=\sum_{(j\notin A\cup H)} P_{ij}^{(k+1)}({}_H f_{jA}^{(k+1,n+1)}(t)),\ \ k=1,2,\cdots,n-1$.

Proof: $_H f_{iA}^{(k,n+1)}(t)=\sum_{(j\notin A\cup H)} ({}_H f_{jA}^{(k+1,n+1)}(t))\sum_z \pi_{k+1}(z,j)\varphi_k(i,z)$

$=\sum_{(j\notin A\cup H)} ({}_H f_{jA}^{(k+1,n+1)}(t))\sum_z P_{zj}^{(k+1)} P({}_H\sigma_A\ge\tau_{k+1}, X(\tau_{k+1}^-)=z)|\ X(\tau_k)=i)$

$=\sum_{(j\notin A\cup H)} ({}_H f_{jA}^{(k+1,n+1)}(t))\sum_z P_{zj}^{(k+1)} P(X(\tau_{k+1}^-)=z)|\ X(\tau_k)=i)$

$=\sum_{(j\notin A\cup H)} ({}_H f_{jA}^{(k+1,n+1)}(t))\sum_z P_{zj}^{(k+1)} P(X(\tau_k)=z)|\ X(\tau_k)=i)$

$=\sum_{(j\notin A\cup H)} ({}_H f_{jA}^{(k+1,n+1)}(t))\sum_z P_{zj}^{(k+1)}\cdot I_{\{z=i\}}=\sum_{(j\notin A\cup H)} P_{ij}^{(k+1)}({}_H f_{jA}^{(k+1,n+1)}(t))$.

For $k=1,2,\cdots, p=1,2,\cdots$, and $\lambda\ge 0$, we let

$Q_{ij}^{*(k)}(t)=P(X(\tau_k)=j,\tau_k\le t|X(\tau_{k-1})=i)$, $\hat{Q}_{ij}^{*(k)}(\lambda)=\int_0^\infty e^{-\lambda t}dQ_{ij}^{*(k)}(t)$,

$\hat{Q}_{ij}^{(k)}(\lambda)=\int_0^\infty e^{-\lambda t}dQ_{ij}^{(k)}(t)$

$\mu_{ij}^{*(k,p)}=\int_0^\infty t^p dQ_{ij}^{*(k)}(t)$, $\mu_{ij}^{(p)}=\int_0^\infty t^p dQ_{ij}^{(1)}(t)$.

Theorem 6 If $X=\{X(t), 0\le t<\tau\}$ is a nonhomogeneous Semi-Markov process, then we have

(a) Letting $z_0=i,\{{}_H f_{iA}^{(n)}(t), i\in E\}$ satisfy the following equations,

$_H f_{iA}^{(1)}(t)=\sum_{j\in A} Q_{ij}^{(1)}(t)$, $_H f_{iA}^{(n+1)}(t)=(\Pi_{k=1}^n\sum_{(z_k\notin A\cup H)} (P_{z_{k-1}z_k}^{(k)})\sum_{j\in A} Q_{z_n j}^{*(n+1)}(t)$

(b) As for $_H f_{iA}(t)$, we have

$_H f_{iA}(t)=\sum_{j\in A} Q_{ij}^{(1)}(t)+\sum_{n=2}^\infty (\Pi_{k=1}^{n-1}\sum_{(z_k\notin A\cup H)} P_{z_{k-1}z_k}^{(k)})\sum_{j\in A} Q_{z_{n-1}j}^{*(n)}(t)$

Proof: $_H f_{iA}^{(1)}(t)=P({}_H\sigma_A\le t, 0<{}_H\sigma_A\le\tau_1|X(0)=i)$

$=P({}_H\sigma_A\le t, {}_H\sigma_A=\tau_1|X(0)=i)=P(X(\tau_1)\in A,\tau_1\le t|X(0)=i)=\sum_{j\in A} Q_{ij}^{(1)}(t)$

$_H f_{iA}^{(n+1)}(t)=\sum_{(j\notin A\cup H)} ({}_H f_{jA}^{(1,n+1)}(t))\sum_z \pi_1(z,j)\cdot\varphi(i,z)$

$=\sum_{(j\notin A\cup H)} ({}_H f_{jA}^{(1,n+1)}(t))\sum_z P_{zj}^{(1)}\cdot I_{\{i=z\}}=\sum_{(j\notin A\cup H)} P_{ij}^{(1)}({}_H f_{jA}^{(1,n+1)}(t))$

Base on Lemma 5, we get the following result by iteration for above equation.

$_H f_{iA}^{(n+1)}(t)=(\Pi_{k=1}^n\sum_{(z_k\notin A\cup H)} P_{z_{k-1}z_k}^{(k)})({}_H f_{z_n A}^{(n,n+1)}(t))$

$=(\Pi_{k=1}^n\sum_{(z_k\notin A\cup H)} P_{z_{k-1}z_k}^{(k)})P({}_H\sigma_A\le t,\tau_n<{}_H\sigma_A\le\tau_{n+1}|\ X(\tau_n)=z_n)$

$=(\Pi_{k=1}^n\sum_{z_k\notin A\cup H} P_{z_{k-1}z_k}^{(k)})P(X(\tau_{n+1})\in A,\tau_{n+1}\le t|\ X(\tau_n)=z_n)$

$=(\Pi_{k=1}^n\sum_{z_k\notin A\cup H} P_{z_{k-1}z_k}^{(k)})\sum_{j\in A} Q_{z_n j}^{*(n+1)}(t)$

While Theorem 5 implies that the result (b) is true.

Theorem 7 If $X=\{X(t), 0\le t<\tau\}$ is a nonhomogeneous Semi-Markov process, then we have

(c) For $n\ge 1$ and $z_0=i,\{{}_H\varphi_{iA}^{(n)}(\lambda), i\in E\}$ satisfy the following equations,

$_H\varphi_{iA}^{(1)}(\lambda)=\sum_{j\in A} Q_{ij}^{(1)}(\lambda)$, $_H\varphi_{iA}^{(n+1)}(t)=(\Pi_{k=1}^n\sum_{(z_k\notin A\cup H)} (P_{z_{k-1}z_k}^{(k)})\sum_{j\in A} Q_{z_n j}^{*(n+1)}(\lambda)$

(d) Then for $_H\varphi_{iA}(t)$, we have

$_H\varphi_{iA}(\lambda)=\sum_{j\in A} Q_{ij}^{(1)}(\lambda)+\sum_{n=2}^\infty (\Pi_{k=1}^{n-1}\sum_{(z_k\notin A\cup H)} (P_{z_{k-1}z_k}^{(k)})\sum_{j\in A} Q_{z_{n-1}j}^{*(n)}(\lambda)$.

Proof: we obtain (c) by taking Laplace-stieltjes transformation about ${}_H f_{iA}^{(1)}(t)$ and ${}_H f_{iA}^{(n+1)}(t)$, then we have the result of (d) immediately by Theorem 6.

Theorem 8 If $X = \{X(t), 0 \le t < \tau\}$ is a nonhomogeneous Semi-Markov process, then we have

(e) For $n \ge 1$ and $z_0 = i, \{{}_H m_{iA}^{n,p}, i \in E, p = 1,2,\cdots\}$ satisfy the following equations,

${}_H m_{iA}^{1,p} = \sum_{j \in A} \mu_{ij}^{(p)}$, ${}_H m_{iA}^{n+1,p} = (\Pi_{k=1}^{n} \sum_{(z_k \notin A \cup H)} P_{z_{k-1} z_k}^{(k)}) \sum_{j \in A} \mu_{z_n j}^{*(n+1,p)}$

(f) Then for ${}_H m_{iA}^{(p)}$, we have ${}_H m_{iA}^{(p)} = \sum_{j \in A} \mu_{ij}^{(p)} + \sum_{n=2}^{\infty} (\Pi_{k=1}^{n-1} \sum_{(z_k \notin A \cup H)} P_{z_{k-1} z_k}^{(k)}) \sum_{j \in A} \mu_{z_{n-1} j}^{*(n,p)}$

Proof: The definition of ${}_H m_{iA}^{(n,p)}$ together with Theorem 6 and Theorem 7 provide us that (e) and (d) are true.

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