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The Complete Convergence of Exchangeable Random Variables

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ABSTRACT

Probability limit theory is one of the branches of probability and also is important basic theory of the science of probability and statistics. Limit theory mainly study independent random variables, but in many practical problems, samples are not independent, or the function of independent sample is not independent, or the verification of independent is more difficult. So the concept of dependent random variables in probability and statistics is mentioned. Exchangeable random variables are a major type of dependent random variable. In this paper we extend the Baum and Katz theorem in the condition of independent, and obtain the specific forms of expression of Baum and Katz theorem in the case of exchangeable random variables.

Key words: Complete convergence; dependent random variables; exchangeable random variables

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INTRODUCTION

If the joint distribution of $X_1, X_2, \cdots, X_n$ is permutation invariant, for each permutation π of $1, 2, \cdots, n$,the joint distribution of $X_1, X_2, \cdots, X_n$ is the same of the joint distribution of $X_{\pi(1)}, X_{\pi(2)}, \cdots, X_{\pi(n)}$,so the finite random variable sequence $X_1, X_2, \cdots, X_n$ is exchangeable. Obviously, the independent and identically distributed random variable sequence is the simplest exchangeable random variable sequence. The concept of exchangeability was first proposed by De Finett[1] in 1930, people using the De Finetti theorem has made some results (see [2]-[3]). In this paper we extend the results of Katz and Baum theorem in the condition of independent and identically distributed random variable sequence to the results of Katz and Baum theorem in the condition of exchangeable random variables. We obtain the Katz and Baum theorem for specific forms of expression in the case of exchangeable random variables.

THE PROBLEM PRESENTION

Since Xu Baolu and Robbins introduced the concept of complete convergence in 1947, many scholars have done a series of research on the complete convergence for independent and identically distributed, results have been studied very perfect (see [4]-[6]), such as Baum and Katz, Bai Zhidong and Su Chun, the most classical results should be Katz and Bau m (1965) the famous theorem in [7]:

Theorem A: suppose $\{X_n; n \ge 1\}$ is the independent and identically distributed random variable sequence, $\alpha > \frac{1}{2}$, $\alpha p > 1$, when $\alpha \le 1$,suppose $EX_1 = 0$,So the two formulas are equivalent:

$$E\left(|X_1|^p\right) < \infty \qquad (1)$$

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le j\le n}\left|S_j\right| \ge \varepsilon n^{\alpha}\right) < \infty, \ \forall \varepsilon > 0 \tag{2}$$

The main result:

Lemma **1** Suppose $\{X_n; n\ge 1\}$ are exchangeable random variables, and the P moments is exist, when $0<p\le 1$ it is an arbitrary stochastic sequence, when $p>1$ it is exchangeable random variables of zero mean .Then for $0<p\le 2$, $E\left(\max_{1\le j\le n}\left|S_j\right|^p\right) \le c\sum_{i=1}^{n} E\left|X_i\right|^p$, when $p>2$, $E\left(\max_{1\le j\le n}\left|S_j\right|^p\right) \le c\left\{\sum_{j=1}^{n} E\left|X_j\right|^p + \left(\sum_{j=1}^{n} EX_j^{\ 2}\right)^{p/2}\right\}$

Proof of lemma in literature [8], 3.1.3. Theorem **1** suppose $\{X_n; n\ge 1\}$ are exchangeable random variables, $Cov\left(f_1(X_1), f_2(X_2)\right) \le 0$. $f_i, i=1,2$ is the type of meaningful and the function does not drop of X_1, X_2, $EX_1=0, \alpha p>1, p<2$, So the two formulas are equivalent:

$$E\left(\left|X_1\right|^p\right) < \infty$$

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le j\le n}\left|S_j\right| \ge \varepsilon n^{\alpha}\right) < \infty, \quad \forall \varepsilon > 0$$

Proof: first we prove(1) implies (2), give q ,satisfy $\left(1+1/\alpha p\right)/2 < q < 1$, take $X_i, i=1,2,\cdots,n$ truncation

$$Y_i(n) = -n^{\alpha q} I_{\left(X_i < -n^{\alpha q}\right)} + X_i I_{\left(\left|X_i\right| \le n^{\alpha q}\right)} + n^{\alpha p} I_{\left(X_i > n^{\alpha q}\right)}$$

$$\tilde{S}_j \triangleq \sum_{i=1}^{j} Y_i(n), \ j=1,2,\cdots,n$$

Because $Y_i(n)$ is monotone non decreasing function, so $Y_1(n), Y_2(n), \cdots, Y_n(n)$ also $Cov\left(f_1\left(Y_1(n)\right), f_2\left(Y_2(n)\right)\right) \le 0$.

$$\left(\max_{1\le j\le n}\left|S_j\right| \ge \varepsilon n^{\alpha}\right) \subset \bigcup_{i=1}^{n}\left(\left|X_i\right| \ge \frac{\varepsilon}{4} n^{\alpha}\right) \cup \bigcup_{1\le i<j\le n}\left(X_i > n^{\alpha q}, X_j > n^{\alpha q}\right)$$

$$\cup\left(X_i < -n^{\alpha q}, X_j < -n^{\alpha q}\right) \cup \left(\max_{1\le j\le n}\left|\tilde{S}_j\right| \ge \frac{\varepsilon}{2} n^{\alpha}\right) \triangleq A_n \cup B_n \cup C_n$$

To prove(2)we just need to prove

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(A_n\right) < \infty, \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(B_n\right) < \infty, \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(C_n\right) < \infty$$

From lemma1 and(1)we obtain

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(A_n\right) \le \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\left|X_1\right| \ge \varepsilon n^{\alpha}\right)$$

$$= \sum_{j=1}^{\infty} \sum_{2^j \le n < 2^{j+1}} n^{\alpha p-1} P(\left|X_1\right| \ge \varepsilon n^{\alpha}) \le \sum_{j=1}^{\infty} 2^{\alpha pj} \sum_{k=j}^{\infty} P\left(\left|X_1\right| \ge \varepsilon 2^{\alpha j}\right)$$

$$= \sum_{j=1}^{\infty} 2^{\alpha pj} \sum_{k=j}^{\infty} P\left(\varepsilon 2^{\alpha k} \le \left|X_1\right| < \varepsilon 2^{\alpha(k+1)}\right)$$

$$=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}2^{\alpha pj}P\left(\varepsilon 2^{\alpha k}\le |X_1|<\varepsilon 2^{\alpha(k+1)}\right)$$

$$\ll \sum_{k=1}^{\infty}2^{\alpha pk}P\left(\varepsilon 2^{k}\le |X_1|^{1/\alpha}<\varepsilon 2^{k+1}\right)\ll E\left(|X_1|^{p}\right)<\infty$$

so $\sum_{n=1}^{\infty}n^{\alpha p-2}P(A_n)<\infty$ is right. Because of the exchangeability and $Cov\left(f_1(X_1),f_2(X_2)\right)\le 0$ the choose of q we know $(1-2q)\alpha p<-1$, so

$$\sum_{n=1}^{\infty}n^{\alpha p-2}P(B_n)\ll \sum_{n=1}^{\infty}n^{\alpha p-2}P^{2}\left(|X_1|>n^{\varepsilon p}\right)$$

$$\le \sum_{n=1}^{\infty}n^{\alpha p-2}n^{-2\alpha pq}l^{-2}\left(n^{q}\right)\left(E\left(|X_1|^{p}\right)\right)^{2}$$

$$\ll \sum_{n=1}^{\infty}n^{(1-2q)\alpha p}n^{-2\alpha pq}<\infty$$

so $\sum_{n=1}^{\infty}n^{\alpha p-2}P(B_n)<\infty$ is right. To prove $\sum_{n=1}^{\infty}n^{\alpha p-2}P(C_n)<\infty$, we first prove $n^{-\alpha}\max_{1\le j\le n}E\left|\tilde{S}_j\right|\to 0\ (n\to\infty)$. Because of $EX_1=0$ and the definition of q we know $\alpha pq>1,\ 1-q>0$, so

$$\begin{aligned}n^{-\alpha}\max_{1\le j\le n}E\left|\tilde{S}_j\right| &\le n^{-\alpha}\sum_{i=1}^{n}\left|EY_i(n)\right|=n^{1-\alpha}\left|E\left(X_1-Y_1(n)\right)\right|\\ &\ll n^{1-\alpha}n^{\alpha q}P\left(|X_1|>n^{\alpha q}\right)\\ &\le n^{1-\alpha+\alpha q}n^{-\alpha pq}E\left(|X_1|^{p}\right)\\ &\ll n^{-(\alpha pq-1)-\alpha(1-q)}\to 0(n\to\infty)\end{aligned}$$

so $n^{-\alpha}\max_{1\le j\le n}E\left|\tilde{S}_j\right|\to 0(n\to\infty)$ is right. From $n^{-\alpha}\max_{1\le j\le n}E\left|\tilde{S}_j\right|\to 0\ (n\to\infty)$ we know $\sum_{n=1}^{\infty}n^{\alpha p-2}P(C_n)<\infty$ is equivalent to

$$\sum_{n=1}^{\infty}n^{\alpha p-2}P\left(\max_{1\le j\le n}\left|\tilde{S}_j-E\tilde{S}_j\right|\ge \varepsilon n^{\alpha}\right)<\infty$$

From lemma **1**,we choose $\lambda>\max\left(2,\frac{\alpha p-1}{\alpha(1-q)+(\alpha pq-1)/2}\right)$,so

$$P\left(\max_{1\le j\le n}\left|\tilde{S}_j-E\tilde{S}_j\right|\ge \varepsilon n^{\alpha}\right)\ll n^{-\alpha\lambda}E\max_{1\le j\le n}\left(\left|\tilde{S}_j-E\tilde{S}_j\right|\right)^{\lambda}$$

$$\ll n^{-\alpha\lambda}\left(nE|Y_1-EY_1|^{\lambda}+\left(nEY_1^{2}\right)^{\lambda/2}\right)$$

$$\ll n^{-\alpha\lambda+1}E|Y_1|^{p}|Y_1|^{\lambda-p}+n^{-\alpha\lambda+\lambda/2}\left(E\left(|X_1|^{p}|X_1|^{2-p}\right)\right)^{\lambda/2}$$

$$\ll n^{-\alpha\lambda+1+\alpha q\lambda-\alpha pq}+n^{-\alpha\lambda+\alpha q\lambda(2-p)/2}$$

so $\sum_{n=1}^{\infty}n^{\alpha p-2}P\left(\max_{1\le j\le n}\left|\tilde{S}_j-E\tilde{S}_j\right|\ge \varepsilon n^{\alpha}\right)$

$$\ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\lambda+1+\alpha q\lambda-\alpha pq} + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\lambda+\lambda/2+\alpha q\lambda-\alpha q\lambda p/2}$$

$$= \sum_{n=1}^{\infty} n^{-1-\alpha(1-q)(\lambda-p)} + \sum_{n=1}^{\infty} n^{\alpha p-2-\lambda(\alpha(1-q)+\lambda((\alpha pq-1)/2))}$$

From the choce of q, λ ,we know

$$-1-\alpha(1-q)(\lambda-p) < -1, \alpha p-2-\lambda\left[\alpha(1-q)+(\alpha pq-1/2)\right] < -1$$

Because the series $\sum_{n=1}^{\infty} n^{-1-\alpha(1-q)(\lambda-p)}$ 和 $\sum_{n=1}^{\infty} n^{\alpha p-2-\lambda(\alpha(1-q)+\lambda((\alpha pq-1)/2))}$ are all convergent, so we obtain

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le j\le n}\left|\tilde{S}_j - E\tilde{S}_j\right| \ge \varepsilon n^{\alpha}\right) < \infty$$

From all of above, we know

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le j\le n}\left|S_j\right| \ge \varepsilon n^{\alpha}\right), \ \forall\varepsilon > 0$$

Next we need to prove(1),for $\max_{1\le j\le n}\left|X_j\right| \le 2\max_{1\le j\le n}\left|S_j\right|$,so because of(2)is right, we obtain

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le j\le n}\left|X_j\right| \ge \varepsilon n^{\alpha}\right) < \infty \ (\forall\varepsilon > 0)$$

From all of above $P\left(\max_{1\le j\le n}\left|X_j\right| \ge \varepsilon n^{\alpha}\right) \to 0$ .Contrary, if not, $\forall\varepsilon > 0, \exists\delta > 0,$ And the natural number sequence n_k , $P\left(\max_{1\le j\le n}\left|X_j\right| \ge \varepsilon 2^{\alpha} n_k^{\ \alpha}\right) > \delta$

It may be well that we suppose $n_{k+1} \ge 2n_k$,Taking into account the $\alpha p > 1$,we obtain

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le j\le n}\left|X_j\right| \ge \varepsilon n^{\alpha}\right) \ge \sum_{k=1}^{\infty} \sum_{n_k\le n\le 2n_k} n^{\alpha p-2} P\left(\max_{1\le j\le n}\left|X_j\right| \ge \varepsilon n^{\alpha}\right)$$
$$\ge \sum_{k=1}^{\infty} n_k^{\ \alpha p-1} P\left(\max_{1\le j\le n_k}\left|X_j\right| \ge \varepsilon 2^{\alpha} n_k^{\ \alpha}\right) \ge \delta\sum_{n=1}^{\infty} n_k^{\ \alpha p-1} = \infty$$

This is will be Contradictory of $\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le j\le n}\left|X_j\right| \ge \varepsilon n^{\alpha}\right) < \infty$ $(\forall\varepsilon > 0)$.

so $P\left(\max_{1\le j\le n}\left|X_j\right| \ge \varepsilon n^{\alpha}\right) \to 0$,because of $P\left(\max_{1\le j\le n}\left|X_j\right| \ge \varepsilon n^{\alpha}\right) \to 0$,we obtain $nP \ge \left(\left|X_1\right| > \varepsilon n^{\alpha}\right) \to 0$

because of the exchangeablility and $Cov\left(f_1(X_1), f_2(X_2)\right) \le 0$ and $e^{-x} \ge 1-x$, $x > 0$ we obtain

$$P\left(\max_{1\le j\le n}\left|X_j\right| \ge \varepsilon n^{\alpha}\right) = P\left(\bigcup_{J=1}^{n}\left(\left|X_j\right| \ge \varepsilon n^{\alpha}\right)\right)$$

$$\ge \sum_{j=1}^{n} P\left(\left|X_j\right| \ge \varepsilon n^{\alpha}\right) - \sum_{1\le i<j\le n} P\left(\left|X_i\right| \ge \varepsilon n^{\alpha}, \left|X_j\right| \ge \varepsilon n^{\alpha}\right)$$

$$\geq nP\left(\left|X_1\right| \geq \varepsilon n^\alpha\right) - n^2 P\left(\left|X_1\right| \geq \varepsilon n^\alpha\right)$$

$$= nP\left(\left|X_1\right| \geq \varepsilon n^\alpha\right)\left(1 - nP\left(\left|X_1\right| \geq \varepsilon n^\alpha\right)\right) \geq \frac{1}{2} nP\left(\left|X_1\right| \geq \varepsilon n^\alpha\right)$$

So $nP\left(\left|X_1\right| \geq \varepsilon n^\alpha\right) \ll P\left(\max_{1\leq j\leq n}\left|X_j\right| \geq \varepsilon n^\alpha\right)$

Because of $\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leq j\leq n}\left|X_j\right| \geq \varepsilon n^\alpha\right) < \infty$ is right, $\forall \varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-1} P\left(\left|X_1\right| \geq \varepsilon n^\alpha\right) < \infty$$

From all of above, we know $\infty > \sum_{n=1}^{\infty} n^{\alpha p-1} P\left(\left|X_1\right| \geq \varepsilon n^\alpha\right) \geq \sum_{j=1}^{\infty} \sum_{2 \leq n < 2^{j+1}} n^{\alpha p-1} P\left(\left|X_1\right| \geq \varepsilon n^\alpha\right)$

$$\geq \sum_{j=1}^{\infty} 2^{\alpha pj} P\left(\left|X_1\right| \geq \varepsilon 2^{\alpha(j+1)} \triangleq \varepsilon_0 2^{\alpha j}\right)$$

$$= \sum_{j=1}^{\infty} 2^{\alpha pj} \sum_{k=j}^{\infty} P\left(\varepsilon_0 2^{\alpha k} \leq \left|X_1\right| \leq \varepsilon_0 2^{\alpha(k+1)}\right)$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{k} 2^{\alpha pj} P\left(\varepsilon_0 2^{\alpha k} \leq \left|X_1\right| \leq \varepsilon_0 2^{\alpha(k+1)}\right)$$

$$\gg \sum_{k=1}^{\infty} 2^{\alpha pj} P\left(\varepsilon_0 2^{\alpha k} \leq \left|X_1\right| \leq \varepsilon_0 2^{\alpha(k+1)}\right)$$

$$\gg E\left(\left|X_1\right|^p\right)$$

So (1)is right.

REFERENCES

[1] B. De Fintti. Funzione Caratteristica Di Unfenomeno Aleatorio. Atti Accad. NaZ. Lincei Rend. cl. *Sci. Fis. Mat. Nat.* vol.4 pp.405-422, **1930**.
[2] J. Blum, H. Chernoff, M. Rosenblatt, H. Teicher. *Canad. Math.* vol.10, **1958**.
[3] R. L. Taylor. Laws of Large Numbers for Dependent Random Variables. Colloquia mathematica societatics Janos Bolyai, 36, Limit Theorems in Probability and Statistics, Veszprem(Hungary), pp.1023-1042, **1982**.
[4] Peng Hanxiang, Dang Xin, Wang Xueqin. *Statistics and Probability Letters*. Vol.27(1), pp.932-938, **2010**.
[5] Levental,S. *Theory Probab* Appl. Vol.45, pp.525-532, **2001**.
[6] Nikolai Delhi Paiva. *Mathematics and Economics*. Vol.42, pp.147-153, **2008**.
[7] Blum L. E, Katz M. *Trars Amer Math Soc*., Vol.120, pp.108-123, **1965**.
[8] Wu Qunying. Probability limit theory of mixing sequences. Science Press, pp.132-133, **2006**.