Available online www.jocpr.com

Journal of Chemical and Pharmaceutical Research, 2014, 6(6):2521-2526

Research Article

ISSN : 0975-7384
CODEN(USA) : JCPRC5

The characterization of moebius sectional curvature of submanifolds on unit Sphere

Nan Ji, Yuxia Tong and Shaohong Yan

College of Sciences, Hebei United University, Tangshan, Hebei, China

---

ABSTRACT

In this paper, let M^m be an m-dimensional submanifold without umbilical point on unit sphere, we prove two Moebius sectional curvature pinching theorems, which give the characterizations of Clifford tori and Veronese submanifolds by the Moebius invariants.

Key words: Moebius submanifold, Moebius second fundamental form, Moebius invariants

---

INTRODUCTION

Since Wang (cf. [1])using conformal differential geometry to establish the theory of conformal differential geometry of submanifolds, and submanifolds are obtained fully invariant system under the conformal group, the conformal differential geometry research has made great progress. Many conformal submanifolds in differential geometry was classified completely (cf [2-6]), which apply the invariant system---Moebius form, Moebius second fundamental form B, Blaschake tensor A, and then submanifold of unit sphere S^n is given a number of important Moebius characters. In this paper, we prove two Moebius sectional curvature pinching theorems, which give the characterizations of Clifford tori and Veronese submanifolds by the Moebius invariants [7].

Orthonormal frame field and Riemannian curvature

Let M be a m-dimentional Riemannian manifold, $e_1, e_2, \cdots, e_m$ a local orthonormal frame field on M, and $\omega_1, \omega_2, \cdots, \omega_m$ is its dual frame field. Then the structure equation of M is given by:

$$d\omega_i = \sum_j \omega_j \wedge \omega_{ji}, \quad \omega_{ij} = -\omega_{ji} \tag{1.1}$$

$$d\omega_{ij} = \sum_l \omega_{il} \wedge \omega_{lj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l \tag{1.2}$$

where ω_{ij} is the Levi-civita connection and R_{ijkl} the Riemannian curvature tensor of M. Ricci tensor R_{ij} and scalar curvature are defined respectively by:

$$R_{ij} := \sum_k R_{kikj}, r := \sum_k R_{kk} \tag{1.3}$$

MOEBIUS INVARIANTS

Let R_1^{n+2} is a n+2-dimentional Lorentzian space, $X=(x_0,x_1,....,x_{n+1})$, $Y=(y_0,y_1,.....,y_{n+1})$, define:

$$<X,Y>=-x_0y_0+x_1y_1+......+x_{n+1}y_{n+1} \tag{2.1}$$

Let $x:M^m\to S^n$ is an Immersed submanifolds without umbilical point on unit sphere, and position vector $Y=\rho(1,x),$

$$\rho^2=\frac{m}{m-1}(\|II\|^2-mH^2) \tag{2.2}$$

then $g=<dY,dY>=\rho^2 dx\cdot dx$ is Moebius invariants.

In the unit sphere, let $\{e_1,e_2,\cdots,e_m\}$ a local orthonormal frame field on M, and $\{\omega_1,\omega_2,\cdots,\omega_m\}$ is its dual frame field. where $1\le i,j,k,l.....\le m$; $m+1\le\alpha,\beta,.....\le m+p=n$, then

A $=\sum_{i,j}A_{ij}\omega_i\wedge\omega_j$, B $=\sum_{i,j,\alpha}B_{ij}^{\alpha}\omega_i\wedge\omega_jE_\alpha$, $\phi=\sum_{i,\alpha}C_i^{\alpha}\omega_iE_\alpha$

where A is Blaschake tensor, B is Moebius form, ϕ is Moebius second fundamental form, then we get the equation as follows：

$$A_{ij,k}-A_{ik,j}=\sum_{\alpha}(B_{ik}^{\alpha}C_j^{\alpha}-B_{ij}^{\alpha}C_k^{\alpha}) \tag{2.3}$$

$$C_{i,j}^{\alpha}-C_{j,i}^{\alpha}=\sum_k(B_{ik}^{\alpha}A_{kj}-B_{kj}^{\alpha}A_{ki}) \tag{2.4}$$

$$B_{ij,k}^{\alpha}-B_{ik,j}^{\alpha}=\delta_{ij}C_k^{\alpha}-\delta_{ik}C_j^{\alpha} \tag{2.5}$$

$$R_{ijkl}=\sum_{\alpha}(B_{ik}^{\alpha}B_{jl}^{\alpha}-B_{il}^{\alpha}B_{jk}^{\alpha})+A_{ik}\delta_{jl}+A_{jl}\delta_{ik}-A_{il}\delta_{jk}-A_{jk}\delta_{il} \tag{2.6}$$

$$\sum_i B_{ii}^{\alpha}=0,\quad trA=\frac{1+m^2R}{2m},\quad \sum_{\alpha}\sum_{i,j}(B_{ij}^{\alpha})^2=\frac{m-1}{m},\quad R=\frac{1}{m(m-1)}\sum_{i,j}R_{ijij} \tag{2.7}$$

Where R_{ijkl} the Riemannian curvature tensor of M, R is is the normal Moebius scalar curvature of M.

PINCHING THEOREMS ABOUT MOEBIUS SECTIONAL CURVATURE

Lemma 1

$$2\sum_{\alpha,\beta}[tr(B_\alpha^2B_\beta^2)-tr(B_\alpha B_\beta)^2]+\sum_{\alpha,\beta}[tr(B_\alpha B_\beta)]^2\le[1+\frac{1}{2}\mathrm{sgn}(p-1)]\|B\|^4 \tag{3.1}$$

if and only if

(i) $p=1$

or (ii) $p=2$, B^{m+1},B^{m+2} at the same time as

$\lambda\overline{B}^{m+1},\mu\overline{B}^{m+2},\lambda^2=\mu^2$,

$$\overline{B}^{m+1} = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 1 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ .. & .. & .. & ... & .. \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \quad \overline{B}^{m+2} = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & -1 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ .. & .. & .. & ... & .. \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}$$

Lemma 2

Let $x: M^m \to S$ is submainfolds without umbilical point on S^n, $\forall a \in R^1$

$$0 = \frac{1}{2}\Delta\|B\|^2 = \|\nabla B\|^2 + (1+a)\sum_{i,j,\alpha,\ t,k} B_{ij}^{\alpha}(B_{tk}^{\alpha}R_{tijk} + B_{ti}^{\alpha}R_{tkjk}) - \frac{(m-1)a}{m}trA$$
$$+ a\sum_{\alpha} tr(B_{\alpha}\nabla\phi_{\alpha}) - (1-a)\sum_{\alpha,\beta}[tr(B_{\alpha}^2 B_{\beta}^2) - tr(B_{\alpha}B_{\beta})^2] + a\sum_{\alpha,\beta}[tr(B_{\alpha}B_{\beta})]^2 - ma\sum_{\alpha} tr(AB_{\alpha}^2)$$

Theorem 1 Let $x: M^m \to S^n$ $(n = m + p)$ is submainfold without umbilical point in S^n, K is the infimum of the sectional curvature, then:

$$K \le \frac{m-1}{2m^2}[1 + \frac{1}{2}\mathrm{sgn}(p-1) - \frac{1}{p}],$$

if $$K \ge \frac{m-1}{2m^2}[1 + \frac{1}{2}\mathrm{sgn}(p-1) - \frac{1}{p}],$$

$x(M)$ is Moebius equivalent to a Versonese surface in S^4, or is equivalent to Clifford tori in S^{m+1}, $S^k(\sqrt{\frac{k}{m}}) \times S^{m-k}(\sqrt{\frac{m-k}{m}})$ $(1 \le k \le m-1)$

Proof of Theorem 1

Let $x: M^m \to S^n(n = m + p)$ is an submanifolds without umbilical point on unit sphere, K is the infimum of the sectional curvature, In Lemma 1, let $a = 0$, then:

$$0 = \|\nabla B\|^2 + \sum_{i,j,\alpha,\ t,k} B_{ij}^{\alpha}(B_{tk}^{\alpha}R_{tijk} + B_{ti}^{\alpha}R_{tkjk}) - \sum_{\alpha,\beta}[tr(B_{\alpha}^2 B_{\beta}^2) - tr(B_{\alpha}B_{\beta})^2] \tag{3.2}$$

Because

$$\sum_{i,j,\alpha,\ t,k} B_{ij}^{\alpha}(B_{tk}^{\alpha}R_{tijk} + B_{ti}^{\alpha}R_{tkjk}) \ge mK\|B\|^2, \quad \frac{\|B\|^4}{p} \le \sum_{\alpha,\beta}[tr(B_{\alpha}B_{\beta})]^2 \le \|B\|^4 \tag{3.3}$$

then $0 \ge \|\nabla B\|^2 + mK\|B\|^2$

$$-\frac{1}{2}\{2\sum_{\alpha,\beta}[tr(B_{\alpha}^2 B_{\beta}^2) - tr(B_{\alpha}B_{\beta})^2] + \sum_{\alpha,\beta}[tr(B_{\alpha}B_{\beta})]^2\} + \frac{1}{2}\sum_{\alpha,\beta}[tr(B_{\alpha}B_{\beta})]^2$$
$$\ge \|\nabla B\|^2 + mK\|B\|^2 - \frac{1}{2}[1 + \frac{1}{2}\mathrm{sgn}(p-1)]\|B\|^4 + \frac{1}{2p}\|B\|^4 \tag{3.4}$$

Because $\|\nabla B\|^2 \ge 0$

Then

$$K-\frac{m-1}{2m^2}[1+\frac{1}{2}\mathrm{sgn}(p-1)-\frac{1}{p}]\le 0 \quad (3.5)$$

if $K\ge\frac{m-1}{2m^2}[1+\frac{1}{2}\mathrm{sgn}(p-1)-\frac{1}{p}]$, then

$$K=\frac{m-1}{2m^2}[1+\frac{1}{2}\mathrm{sgn}(p-1)-\frac{1}{p}] \quad (3.6)$$

Because $\|\nabla B\|=0 \quad \phi=0$, according to the lemma 2, we get the following:

(i) $p=1$, $K=0$, $x(M)$ is Moebius equivalent to a Clifford minimal tori $S^k(\sqrt{\frac{k}{m}})\times S^{m-k}(\sqrt{\frac{m-k}{m}})$ () $1\le k\le m-1$ in S^{m+1}

(ii) $p=2$, $K=1/8$, $x(M)$ is Moebius equivalent to a Versonese surface in S^4

Theorem 2 Let $\mathrm{x}:\mathrm{M}^{\mathrm{m}}\to\mathrm{S}^{\mathrm{n}}\ (n=m+p)$ is submainfolds without umbilical point in S^n , K is infimum of the sectional curvature, $D=A-\frac{1}{m}trA\cdot id$, then:

$$K\le\frac{m-2}{2(m+1)}\sqrt{\frac{m}{m-1}}\|D\|+\frac{1}{m+1}trA ,$$

if $K\ge\frac{m-2}{2(m+1)}\sqrt{\frac{m}{m-1}}\|D\|+\frac{1}{m+1}trA$

$x(M)$ is Moebius equivalent to a Versonese surface $S^m(\sqrt{\frac{2(m+1)}{m}})$.

Proof of Theorem 2

Let $x:M^m\to S^n(n=m+p$) is an submanifolds without umbilical point on unit sphere, K is the infimum of the sectional curvature, in Lemma 2, let $a=m/(m+2)$, from (3.3) and

$$\left|trDB^2\right|\le\frac{m-2}{\sqrt{m(m-1)}}\|D\|$$

We obtain

$$\frac{2(m+1)m}{m+2}K-\frac{m}{m+2}\cdot\frac{m(m-2)}{\sqrt{m(m-1)}}\|D\|-\frac{2m}{m+2}trA\le 0 \quad (3.7)$$

Then

$$K\le\frac{m-2}{2(m+1)}\sqrt{\frac{m}{m-1}}\|D\|+\frac{1}{m+1}trA \quad (3.8)$$

If $K\ge\frac{m-2}{2(m+1)}\sqrt{\frac{m}{m-1}}\|D\|+\frac{1}{m+1}trA$

Then

$$K = \frac{m-2}{2(m+1)}\sqrt{\frac{m}{m-1}}\|D\| + \frac{1}{m+1}trA \tag{3.9}$$

From $\|\nabla B\| = 0$ we obtain $\phi = 0$, Let $\{e_1, e_2, \cdots, e_m\}$ is a local standard orthogonal basis on TM, $A_{ij} = \lambda_i \delta_{ij}$, $D_{ij} = \overline{\lambda}_i \delta_{ij}$.

From (3.7) take the equal sign, get all the λ_i is equal to each other, let $\lambda_2 = ... = \lambda_m$, then $\tilde{\lambda}_2 = ... = \tilde{\lambda}_m$. Then, we let:

$$B_{11} = -(m-1)\mu, \quad B_{22} = ... = B_{mm} = \mu, \quad \tilde{\lambda}_1 = \frac{m-1}{m}(\lambda_1 - \lambda), \quad \tilde{\lambda}_2 = ... = \tilde{\lambda}_m = \frac{1}{m}(\lambda - \lambda_1)$$

$$\|D\| = \sqrt{\sum_i \tilde{\lambda}_i^2} = \frac{\sqrt{m(m-1)}}{m}|\lambda - \lambda_1| \tag{3.10}$$

from (2.6), we get

$$(m-1)^2\mu^2 + (m-1)\mu^2 = \frac{m-1}{m}, \quad \mu = \pm\frac{1}{m} \tag{3.11}$$

when $\alpha \geq 2$, $B_{1\alpha} = 0$

because $B_{11} \neq B_{\alpha\alpha}$, we obtain

$$\omega_{\alpha 1} = 0$$

$$\sum_j B_{1\alpha,j}\omega_j = dB_{1\alpha} + \sum_j (B_{1j}\omega_{j\alpha} + B_{j\alpha}\omega_{j1}) = (B_{\alpha\alpha} - B_{11})\omega_{\alpha 1} = 0 \tag{3.12}$$

$$-\frac{1}{2}\sum_{k,l} R_{1\alpha kl}\omega_k \wedge \omega_l = d\omega_{1\alpha} - \sum_k \omega_{1k} \wedge \omega_{k\alpha} = 0 \tag{3.13}$$

From (2.5)

$$0 = R_{1\alpha 1\alpha} = B_{11}B_{\alpha\alpha} + A_{11} + A_{\alpha\alpha} = -(m-1)\mu^2 + \lambda_1 + \lambda \tag{3.14}$$

$$\lambda_1 + \lambda = \frac{m-1}{m^2} \tag{3.15}$$

$$0 = \sum_j A_{1\alpha,j}\omega_j = dA_{1\alpha} + \sum_j A_{1j}\omega_{\alpha j} + \sum_j A_{\alpha j}\omega_{j1} = dA_{1\alpha} + (A_{11} - A_{\alpha\alpha})\omega_{1\alpha} \tag{3.16}$$

Then $A_{1\alpha,j} = A_{1j,\alpha} = 0$ $\quad A_{11,\alpha} = 0$, $A_{1\alpha,\alpha} = A_{\alpha\alpha,1} = 0$

$$A_{11,1}\omega_1 + \sum_\alpha A_{11,\alpha}\omega_\alpha = \sum_j A_{11,j}\omega_j \qquad (2 \leq \alpha \leq m, 1 \leq j \leq m) \tag{3.17}$$

Because of $0 = \sum_j A_{11,j}\omega_j = dA_{11} + \sum_j A_{1j}\omega_{1j} + \sum_j A_{j1}\omega_{j1} = dA_{11}$, we get $\lambda_1 = const$

if $\lambda \neq \lambda_1$ and $\omega_1 = \omega_2 = ... = \omega_m = 0$, assuming that $M^m = M_1^1 \times M_2^{m-1}$, K=0

From (3.9), we obtain $K=\frac{m-1}{2m(m+1)}$, which is contradiction with K=0, then $\lambda_1=\lambda,\tilde{\lambda}_1=...=\tilde{\lambda}_m=0$, D=0.

From (3.9), we obtain R=K=$1/[m(m+1)]$. $x(M)$ is Moebius equivalent to minimal submanifold in S^n, then $\rho^2=const$, $A_{ij}=\delta_{ij}/2$.

From $g=\rho^2 dx\cdot dx$, we get $K=\rho^{-2}K_E$ and $\rho^{-2}=2tr(A)/m$.

From $trA=(1+m^2R)/2m$, we get $\rho^{-2}=(1/m^2)+R$, $K_E=\frac{m}{2(m+1)}$,

Then we obtain $\overline{M}$ is isometric to the Veronese surface $S^m(\sqrt{2(m+1)/m})$.

And $x(M)$ is Moebius equivalent to a Versonese surface $x_m:S^m(\sqrt{\frac{2(m+1)}{m}})\to S^{m+p}, p=\frac{1}{2}(m-1)(m-2)$

Acknowledgment
This work is supported by the natural science foundation of Hebei Province (No.A2013209278), and the Scientific Technology Research and Development Plan Project of Tang Shan City (No. 12140208A-7).

REFERENCES

[1] Wang.C.P. *Manuscripta Math*, **1998**, 96, 517-534.
[2] Li H Z, Wang C P, Wu F E. *Math Ann*, **2001**, 319, 707-714.
[3] Li H Z, liu H L, Wang C P et al. *Acta Mathematica Sinica, English Series*, **2002**, 18(3), 437-446.
[4] Zhang B.; Zhang S.; Lu G.. *Journal of Chemical and Pharmaceutical Research*, **2013**, 5(9), 256-262.
[5] Zhang B.; *International Journal of Applied Mathematics and Statistics*, **2013**, 44(14), 422-430.
[6] Zhang B.; Yue H.. *International Journal of Applied Mathematics and Statistics*, **2013**, 40(10), 469-476.
[7] Zhang B.; Feng Y.. *International Journal of Applied Mathematics and Statistics*, **2013**, 40(10), 136-143.