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Some new sufficient conditions for generalized strictly diagonally dominant matrices

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ABSTRACT

Generalized strictly diagonally dominant matrices have wide applications in science and engineering, but it is very difficult to determine whether a given matrix is a generalized strictly diagonally dominant matrix or not in practice. In this paper, we give several practical conditions for generalized strictly diagonally dominant matrices by constructing different positive diagonal matrix and applying some techniques of inequalities, which improve and generalize some existing conclusions. Effectiveness of results is illustrated by some numerical examples.

Keywords: Positive diagonal matrix; Diagonal dominant; generalized strictly diagonally dominant matrices; Irreducible matrix; nonzero elements chain

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INTRODUCTION

Generalized strictly diagonally dominant matrices is a special class matrix with widely use in the engineering, which is not only one of the important research topics in computational mathematics and matrix theory, but also has important practical value in control theory, elasticity, mathematical economics, the power system theory and other fields. However, it is difficult to determine generalized strictly diagonally dominant matrices in practice. The problem is investigated in some papers, e.g., see [1-15]. This paper gives several new conditions for generalized strictly diagonally dominant matrices, which improve some related results, and gives some corresponding numerical examples to illustrate the validity of the results.

Denote $C^{n\times n}$ as the set of complex $n\times n$ matrix and $N=\{1,2,\cdots,n\}$. Let $\mathbf{A}=(a_{ij})\in C^{n\times n}$, $R_i(\mathbf{A})=\sum_{j\neq i}|a_{ij}|(i\in N)$; $N_1=\{i\in N|0<|a_{ii}|=R_i(\mathbf{A})\}$; $N_2=\{i\in N|0<|a_{ii}|<R_i(\mathbf{A})\}$; $N_3=\{i\in N||a_{ii}|>R_i(\mathbf{A})\}$. If $|a_{ii}|>R_i(A)(i\in N)$, then A is said to be a strictly diagonally dominant matrix and is denoted by $A\in D$. If there exists a positive diagonal matrix $\mathbf{X}$ such that $AX\in D$, then A is said to be a generalized strictly diagonally dominant matrix and is denoted by $A\in \overline{D}$.

Obviously, $N_1\cap N_2=\phi(\text{empty set})$; $N_1\cap N_3=\phi$; $N_2\cap N_3=\phi$; $N_1\cup N_2\cup N_3=N$. We take $\sum_{t\in\phi}\bullet=0$. If $N_1\cup N_2=\phi$, then $A\in D$. If $A\in\overline{D}$, then the all diagonal entries of $\mathbf{A}$ are non-zero and exists at least one strict diagonally dominant row (see [12]), that is $N_3\neq\phi$. So we always assume that $N_1\cup N_2$ and N_3 are not empty set.

Definition 1[10, 13] A matrix $A=(a_{ij})\in C^{n\times n}$ is called irreducibly diagonally dominant if A is irreducible,

$|a_{ii}| \geq R_i(A)(i \in \mathrm{N})$ and a strict inequality holds for at least one $i \in \mathrm{N}$.

Definition 2[13] $\boldsymbol{A} = (a_{ij}) \in \mathrm{C}^{n\times n}$ is called a diagonally dominant matrix with nonzero elements chain if $|a_{ii}| \geq R_i(\mathbf{A})$ ($i \in \mathrm{N}$) and at least one strict inequality holds every i with $|a_{ii}| = R_i(\mathbf{A})$ there exists a nonzero elements chain $a_{ij_1}a_{j_1j_2}\cdots a_{j_{k-1}j_k} \neq 0$ such that $|a_{j_kj_k}| > R_{j_k}(\mathbf{A})$.

Lemma 1[13] Let $\boldsymbol{A} = (a_{ij}) \in \mathbf{C}^{n\times n}$ is an irreducibly diagonally dominant matrix, then $\boldsymbol{A} \in \overline{\mathrm{D}}$.

Lemma 2[13] Let $\boldsymbol{A} = (a_{ij}) \in \mathbf{C}^{n\times n}$ is a diagonally dominant matrix with nonzero elements chain, then $\boldsymbol{A} \in \overline{\mathrm{D}}$.

Lemma 3[15] Let $\boldsymbol{A} = (a_{ij}) \in \mathrm{C}^{n\times n}$, if there exists positive diagonal matrix $\boldsymbol{X}$, such that $\boldsymbol{AX} \in \overline{\mathrm{D}}$, then $\boldsymbol{A} \in \overline{\mathrm{D}}$.

The paper [3] gives the following main result: Theorem1. Let $\boldsymbol{A} = (a_{ij}) \in \mathrm{C}^{n\times n}$. If

$$|a_{ii}| > \frac{R_i(\boldsymbol{A})}{R_i(\boldsymbol{A}) - |a_{ii}|}[\sum_{t\in \mathrm{N}_1}|a_{it}| + \sum_{t\in \mathrm{N}_2-\{i\}}|a_{it}|\frac{R_t(\boldsymbol{A}) - |a_{tt}|}{R_t(\boldsymbol{A})} + \sum_{t\in \mathrm{N}_3}|a_{it}|\frac{P_t}{|a_{tt}|}] \qquad (\forall i \in \mathrm{N}_2)$$

And $|a_{ii}| \neq \sum_{t\in \mathrm{N}_1-\{i\}}|a_{it}|$ $(i \in \mathrm{N}_1)$, where

$$r = \max_{i\in \mathrm{N}_3}\left(\frac{\sum_{t\in \mathrm{N}_1}|a_{it}| + \sum_{t\in \mathrm{N}_2}|a_{it}|}{|a_{ii}| - \sum_{t\in \mathrm{N}_3-\{i\}}|a_{it}|}\right),\ P_i = \sum_{t\in \mathrm{N}_1}|a_{it}| + \sum_{t\in \mathrm{N}_2}|a_{it}| + r\sum_{t\in \mathrm{N}_3-\{i\}}|a_{it}|\,(i \in N_3) \qquad (1)$$

Then $\boldsymbol{A}$ is a generalized strictly diagonally dominant matrices.

1. THE MAIN CONCLUSION

In order to facilitate the description, we will further introduce the following marks: **let** $\boldsymbol{A} = (a_{ij}) \in \mathrm{C}^{n\times n}$

$$x_i = \frac{R_i(\mathbf{A}) - |a_{ii}|}{R_i(\mathbf{A})}(\forall i \in \mathrm{N}_2); \quad x_i = \frac{P_i}{|a_{ii}|}(\forall i \in \mathrm{N}_3);$$

Where P_i as type (1), and

$$M_i = \frac{\sum_{t\in \mathrm{N}_1}|a_{it}| + \sum_{t\in \mathrm{N}_2}|a_{it}|x_t}{P_i - \sum_{t\in \mathrm{N}_3-\{i\}}|a_{it}|x_t}(\forall i \in \mathrm{N}_3); \quad K_i = \frac{\sum_{t\in \mathrm{N}_2}|a_{it}|x_t + \sum_{t\in \mathrm{N}_3}|a_{it}|x_t}{|a_{ii}| - \sum_{t\in \mathrm{N}_1-\{i\}}|a_{it}|}(\forall i \in \mathrm{N}_1);$$

$$\lambda = \max\{\max_{i\in \mathrm{N}_3} M_i, \max_{i\in \mathrm{N}_1} K_i\}$$

When $\mathrm{N}_1 \neq \phi$ and $|a_{ii}| = \sum_{t\in \mathrm{N}_1-\{i\}}|a_{it}|$, take $K_i = 1(\forall i \in \mathrm{N}_1)$; and when $\mathrm{N}_1 = \phi$, take $K_i = 0$.

Theorem2. Let $\boldsymbol{A} = (a_{ij}) \in \mathrm{C}^{n\times n}$. If

$$|a_{ii}|x_i > \sum_{t\in \mathrm{N}_2-\{i\}}|a_{it}|x_t + \lambda(\sum_{t\in \mathrm{N}_1}|a_{it}| + \sum_{t\in \mathrm{N}_3}|a_{it}|x_t)(\forall i \in \mathrm{N}_2) \qquad (2)$$

And $|a_{ii}| \neq \sum_{t\in \mathrm{N}_1-\{i\}}|a_{it}|$ $(\forall i \in \mathrm{N}_1)$, then $\boldsymbol{A} \in \overline{\mathrm{D}}$.

Proof. By the expression of r, we know that $0 \leq r < 1$ and

$$r|a_{ii}| \geq \sum_{t\in \mathrm{N}_1}|a_{it}| + \sum_{t\in \mathrm{N}_2}|a_{it}| + r\sum_{t\in \mathrm{N}_3-\{i\}}|a_{it}| = P_i \qquad (\forall i \in \mathrm{N}_3)$$

Which gives $0 \leq x_i \leq r < 1$ $(\forall i \in \mathrm{N}_3)$.

When $N_1 \neq \phi$ and $T = \{i \mid \Lambda_i = \sum_{t\in N_1}|a_{it}|, i\in N_3\} = \phi$, let

$$m_i = \frac{|a_{ii}|x_i - \sum_{t\in N_2-\{i\}}|a_{it}|x_t}{\sum_{t\in N_1}|a_{it}| + \sum_{t\in N_3}|a_{it}|x_t} \quad (\forall i\in N_2)$$

If $\sum_{t\in N_1}|a_{it}| + \sum_{t\in N_3}|a_{it}|x_t = 0$, set $m_i = +\infty$. By $|a_{ii}| \neq \sum_{t\in N_1-\{i\}}|a_{it}|$ $(\forall i\in N_1)$, $0 < x_i < 1$ $(\forall i\in N_2)$, and the expression of M_i and K_i, we get $0 \le M_i < 1 (i\in N_3)$ and $0 \le K_i < 1 (i\in N_1)$, which gives $0 \le \lambda < 1$. So there must exist a positive number δ, such that

$$\lambda < \delta < \min\{\frac{1}{\max_{i\in N_3} x_i}, \min_{i\in N_2} m_i, 1\}$$

By $\delta > \max_{i\in N_1} K_i$, $\delta < \min_{i\in N_2} m_i$, $\delta > \max_{i\in N_3} M_i$ and $\delta x_i < 1$ $(\forall i\in N_3)$, we get

$$\delta \sum_{t\in N_1-\{i\}}|a_{it}| + \sum_{t\in N_2}|a_{it}|x_t + \sum_{t\in N_3}|a_{it}|x_t < \delta|a_{ii}| \quad (\forall i\in N_1)$$

$$\delta\sum_{t\in N_1}|a_{it}| + \sum_{t\in N_2-\{i\}}|a_{it}|x_t + \delta\sum_{t\in N_3}|a_{it}|x_t < |a_{ii}|x_i \quad (\forall i\in N_2)$$

$$\sum_{t\in N_1}|a_{it}| + \sum_{t\in N_2}|a_{it}|x_t + \delta\sum_{t\in N_3-\{i\}}|a_{it}|x_t < \delta P_i \quad (\forall i\in N_3) \qquad (3)$$

Hence, there always exists a sufficient small positive number ε, such that

$$\delta \sum_{t\in N_1-\{i\}}|a_{it}| + \sum_{t\in N_2}|a_{it}|x_t + \sum_{t\in N_3}|a_{it}|(x_t+\varepsilon) < \delta|a_{ii}| \quad (\forall i\in N_1) \qquad (4)$$

$$\delta\sum_{t\in N_1}|a_{it}| + \sum_{t\in N_2-\{i\}}|a_{it}|x_t + \delta\sum_{t\in N_3}|a_{it}|(x_t+\varepsilon) < |a_{ii}|x_i \quad (\forall i\in N_2) \qquad (5)$$

$$\delta(x_i+\varepsilon) < 1 \quad (\forall i\in N_3)$$

Constructing positive diagonal matrix $\boldsymbol{D} = \mathrm{diag}\{d_1, d_2, \cdots, d_n\}$, and let $\boldsymbol{B} = \boldsymbol{AD} = (b_{ij})_{n\times n}$, where

$$\begin{cases} d_i = \delta (i\in N_1) \\ d_i = x_i (i\in N_2) \\ d_i = \delta(x_i+\varepsilon)(i\in N_3) \end{cases}$$

For $\forall i\in N_1$, by (4), we have

$$R_i(\boldsymbol{B}) = \sum_{j\neq i}|b_{ij}| = \delta\sum_{t\in N_1-\{i\}}|a_{it}| + \sum_{t\in N_2}|a_{it}|x_t + \sum_{t\in N_3}|a_{it}|\delta(x_t+\varepsilon) < \delta|a_{ii}| = |b_{ii}|$$

For $\forall i\in N_2$, by (5), we have

$$R_i(\boldsymbol{B}) = \sum_{j\neq i}|b_{ij}| = \delta\sum_{t\in N_1}|a_{it}| + \sum_{t\in N_2-\{i\}}|a_{it}|x_t + \delta\sum_{t\in N_3}|a_{it}|(x_t+\varepsilon) < |a_{ii}|x_i = |b_{ii}|$$

For $\forall i\in N_3$, by (3), we have

$$\begin{aligned} R_i(\boldsymbol{B}) = \sum_{j\neq i}|b_{ij}| &= \delta\sum_{t\in N_1}|a_{it}| + \sum_{t\in N_2}|a_{it}|x_t + \sum_{t\in N_3-\{i\}}|a_{it}|\delta(x_t+\varepsilon) \\ &\le \sum_{t\in N_1}|a_{it}| + \sum_{t\in N_2}|a_{it}|x_t + \delta\sum_{t\in N_3-\{i\}}|a_{it}|x_t + \delta\varepsilon\sum_{t\in N_3-\{i\}}|a_{it}|x_t \\ &< \delta P_i + \delta\varepsilon|a_{ii}| = |a_{ii}|d_i = |b_{ii}| \end{aligned}$$

When $N_1 \neq \phi$ and $T = \{i \mid \Lambda_i = \sum_{t\in N_1}|a_{it}|, i\in N_3\} \neq \phi$, by $|a_{ii}| \neq \sum_{t\in N_1-\{i\}}|a_{it}|$ $(\forall i\in N_1)$ and the expression of K_i and M_i, we get $0 \le K_i < 1$ $(\forall i\in N_1)$ and $\max_{i\in N_3} M_i = 1$, which gives $\lambda = 1$. So there must exist a positive number δ, such that

$$1 = \lambda < \delta < \min\{\frac{1}{\max_{i\in N_3} x_i}, \min_{i\in N_2} m_i\}$$

Hence, there always exists a sufficient small positive number ε, such that $\delta x_i + \varepsilon < 1\ (\forall i \in \mathrm{N}_3)$ and $\delta + \varepsilon < \min_{i\in \mathrm{N}_2} m_i\ \ (\forall i \in \mathrm{N}_2)$. Constructing positive diagonal matrix $\boldsymbol{D} = \mathrm{diag}\{d_1, d_2, \cdots, d_n\}$ and let $\boldsymbol{B} = \boldsymbol{AD} = (b_{ij})_{n\times n}$, where

$$\begin{cases} d_i = \delta (i \in N_1) \\ d_i = x_i (i \in N_2) \\ d_i = \delta x_i + \varepsilon (i \in N_3) \end{cases}$$

For $\forall i \in \mathrm{N}_1$, by $|a_{ii}| \neq \sum_{t\in \mathrm{N}_1-\{i\}} |a_{it}|\ (\forall i \in \mathrm{N}_1)$, $d_i < 1\ (i \in \mathrm{N}_3 \bigcup \mathrm{N}_2)$ and $1 < \delta$, we get

$$\begin{aligned} R_i(\boldsymbol{B}) = \sum_{j\neq i} |b_{ij}| &= \delta \sum_{t\in \mathrm{N}_1-\{i\}} |a_{it}| + \sum_{t\in \mathrm{N}_2} |a_{it}| x_t + \sum_{t\in \mathrm{N}_3} |a_{it}| (\delta x_t + \varepsilon) \\ &< \delta \sum_{t\in \mathrm{N}_1-\{i\}} |a_{it}| + \sum_{t\in \mathrm{N}_2} |a_{it}| + \sum_{t\in \mathrm{N}_3} |a_{it}| \\ &< \delta (\sum_{t\in \mathrm{N}_1-\{i\}} |a_{it}| + \sum_{t\in \mathrm{N}_2} |a_{it}| + \sum_{t\in \mathrm{N}_3} |a_{it}|) = \delta |a_{ii}| = |b_{ii}| \end{aligned}$$

For $\forall i \in \mathrm{N}_2$, by $\delta + \varepsilon < \min_{i\in \mathrm{N}_2} m_i$, we get

$$(\delta + \varepsilon) \sum_{t\in \mathrm{N}_1} |a_{it}| + (\delta + \varepsilon) \sum_{t\in \mathrm{N}_3} |a_{it}| x_t + \sum_{t\in \mathrm{N}_2-\{i\}} |a_{it}| x_t < |a_{ii}| x_i \quad (\forall i \in \mathrm{N}_2)$$

Therefore

$$\begin{aligned} R_i(\mathbf{B}) = \sum_{j\neq i} |b_{ij}| &= \delta \sum_{t\in \mathrm{N}_1} |a_{it}| + \sum_{t\in \mathrm{N}_2-\{i\}} |a_{it}| x_t + \sum_{t\in \mathrm{N}_3} |a_{it}| (\delta x_t + \varepsilon) \\ &< (\delta + \varepsilon) \sum_{t\in \mathrm{N}_1} |a_{it}| + (\delta + \varepsilon) \sum_{t\in \mathrm{N}_3} |a_{it}| x_t + \sum_{t\in \mathrm{N}_2-\{i\}} |a_{it}| x_t \\ &< |a_{ii}| x_i = |b_{ii}| \end{aligned}$$

For $\forall i \in \mathrm{N}_3$, by $\delta > 1$, $0 \leq x_i \leq r < 1\ (\forall i \in \mathrm{N}_3)$ and $0 < x_i < 1\ (\forall i \in \mathrm{N}_2)$, we get

$$\begin{aligned} R_i(\mathbf{B}) = \sum_{j\neq i} |b_{ij}| &= \delta \sum_{t\in \mathrm{N}_1} |a_{it}| + \sum_{t\in \mathrm{N}_2} |a_{it}| x_t + \sum_{t\in \mathrm{N}_3-\{i\}} |a_{it}| (\delta x_t + \varepsilon) \\ &\leq \delta (\sum_{t\in \mathrm{N}_1} |a_{it}| + \sum_{t\in \mathrm{N}_2} |a_{it}| + r \sum_{t\in \mathrm{N}_3-\{i\}} |a_{it}|) + \varepsilon \sum_{t\in \mathrm{N}_3-\{i\}} |a_{it}| \\ &< \delta P_i + \varepsilon |a_{ii}| = |a_{ii}| d_i = |b_{ii}| \end{aligned}$$

Thus, when $\mathrm{N}_1 \neq \phi$ we have proved that $|b_{ii}| > R_i(\boldsymbol{B})\ (\forall i \in \mathrm{N})$, i.e. $\boldsymbol{B} \in \mathrm{D}$. Therefore, $\boldsymbol{A} \in \overline{\mathrm{D}}$.

When $\mathrm{N}_1 = \phi$, at this time $\lambda = \max_{i\in N_3} M_i$, type (3) into

$$|a_{ii}| x_i > \sum_{t\in \mathrm{N}_2-\{i\}} |a_{it}| x_t + \lambda \sum_{t\in \mathrm{N}_3} |a_{it}| x_t (\forall i \in \mathrm{N}_2)$$

Let

$$m_i = \frac{|a_{ii}| x_i - \sum_{t\in \mathrm{N}_2-\{i\}} |a_{it}| x_t}{\sum_{t\in \mathrm{N}_3} |a_{it}| x_t}\ (\forall i \in \mathrm{N}_2)$$

If $\sum_{t\in \mathrm{N}_3} |a_{it}| x_t = 0$, set $m_i = +\infty$. By the expression of M_i, we get $0 \leq M_i < 1 (i \in N_3)$ and $0 \leq \lambda < 1$. So there must exist a positive number δ, such that

$$\lambda < \delta < \min\{\frac{1}{\max_{i\in \mathrm{N}_3} x_i}, \min_{i\in \mathrm{N}_2} m_i, 1\}$$

By $\delta < \min_{i\in \mathrm{N}_2} m_i$, $\delta > \max_{i\in \mathrm{N}_3} M_i$ and $\delta x_i < 1\ (\forall i \in \mathrm{N}_3)$, we get

$$\sum_{t\in N_2-\{i\}}|a_{it}|x_t+\delta\sum_{t\in N_3}|a_{it}|x_t<|a_{ii}|x_i\ (\forall i\in N_2)$$
$$\sum_{t\in N_2}|a_{it}|x_t+\delta\sum_{t\in N_3-\{i\}}|a_{it}|x_t<\delta P_i\ (\forall i\in N_3) \tag{6}$$

Hence, there always exists a sufficient small positive number ε , such that

$$\sum_{t\in N_2-\{i\}}|a_{it}|x_t+\delta\sum_{t\in N_3}|a_{it}|(x_t+\varepsilon)<|a_{ii}|x_i\ (\forall i\in N_2) \tag{7}$$
$$\delta(x_i+\varepsilon)<1\ (\forall i\in N_3)$$

Constructing positive diagonal matrix $\boldsymbol{D}=\mathrm{diag}\{d_1,d_2,\cdots,d_n\}$, and

$$d_i=\begin{cases}x_i & (i\in N_2)\\ \delta(x_i+\varepsilon) & (i\in N_3)\end{cases}$$

Let matrix $\boldsymbol{B}=\boldsymbol{AD}=(b_{ij})_{n\times n}$.

For any $i\in N_2$,by type (7),we can know

$$R_i(\boldsymbol{B})=\sum_{j\neq i}|b_{ij}|=\sum_{t\in N_2-\{i\}}|a_{it}|x_t+\delta\sum_{t\in N_3}|a_{it}|(x_t+\varepsilon)<|a_{ii}|x_i=|b_{ii}|$$

For any $i\in N_3$,by type (6),we can know

$$\begin{aligned}R_i(\boldsymbol{B})&=\sum_{j\neq i}|b_{ij}|=\sum_{t\in N_2}|a_{it}|x_t+\sum_{t\in N_3-\{i\}}|a_{it}|\delta(x_t+\varepsilon)\\&\le\sum_{t\in N_2}|a_{it}|x_t+\delta\sum_{t\in N_3-\{i\}}|a_{it}|x_t+\delta\varepsilon\sum_{t\in N_3-\{i\}}|a_{it}|x_t\\&<\delta P_i+\delta\varepsilon|a_{ii}|=|a_{ii}|d_i=|b_{ii}|\end{aligned}$$

Thus, when $N_1=\phi$ we have proved that $|b_{ii}|>R_i(\boldsymbol{B})\ (\forall i\in N)$, i.e. $\boldsymbol{B}\in D$ .Therefore, $\boldsymbol{A}\in\overline{D}$.
To sum up, the theory is correct.

Remark due to $0\le\lambda\le1$, the theory 2 improve the main conclusion of the paper [1], which expand the scope of determination.

Theorem3. Let $\boldsymbol{A}=(a_{ij})\in\mathbf{C}^{n\times n}$ be irreducible. If

$$|a_{ii}|x_i\ge\sum_{t\in N_2-\{i\}}|a_{it}|x_t+\lambda(\sum_{t\in N_1}|a_{it}|+\sum_{t\in N_3}|a_{it}|x_t)\quad(\forall i\in N_2) \tag{8}$$

And at least one strict inequality holds in (8), then $\boldsymbol{A}\in\overline{D}$.

Proof. By the irreducibility of A, we know that $0<x_i<1(\forall i\in N_2),\ 0<x_i<1,$
$0<M_i\le1(\forall i\in N_3)$, $0<K_i<1(\forall i\in N_1)$ .By the expression of λ ,we know $0<\lambda\le1$ .So there must exist a positive number δ , such that

$$\lambda\le\delta\le\min\{\frac{1}{\max_{i\in N_3}x_i},\min_{i\in N_2}m_i,1\}$$

Constructing positive diagonal matrix $\boldsymbol{D}=\mathrm{diag}\{d_1,d_2,\cdots,d_n\}$, and

$$d_i=\begin{cases}\delta & (i\in N_1)\\ x_i & (i\in N_2)\\ \delta x_i & (i\in N_3)\end{cases}$$

Let matrix $\boldsymbol{B}=\boldsymbol{AD}=(b_{ij})_{n\times n}$, we can know that B is irreducible.

For any $i\in \mathrm{N}_1$, due to $0<\delta\le 1$ and $\delta\ge \max_{i\in \mathrm{N}_1} K_i$,we can get

$$\delta\sum_{t\in \mathrm{N}_1-\{i\}}|a_{it}|+\sum_{t\in \mathrm{N}_2}|a_{it}|x_t+\sum_{t\in \mathrm{N}_3}|a_{it}|x_t\le \delta|a_{ii}|\ (\forall i\in \mathrm{N}_1)$$

Therefore

$$R_i(\boldsymbol{B})=\sum_{j\ne i}|b_{ij}|=\delta\sum_{t\in \mathrm{N}_1-\{i\}}|a_{it}|+\sum_{t\in \mathrm{N}_2}|a_{it}|x_t+\sum_{t\in \mathrm{N}_3}|a_{it}|\delta x_t$$
$$\le \delta\sum_{t\in \mathrm{N}_1-\{i\}}|a_{it}|+\sum_{t\in \mathrm{N}_2}|a_{it}|x_t+\sum_{t\in \mathrm{N}_3}|a_{it}|x_t\le \delta|a_{ii}|=|b_{ii}|$$

For any $i\in \mathrm{N}_2$, since $\delta\le \min_{i\in \mathrm{N}_2} m_i\ (\forall i\in \mathrm{N}_2)$, we can know

$$\delta\sum_{t\in \mathrm{N}_1}|a_{it}|+\delta\sum_{t\in \mathrm{N}_3}|a_{it}|x_t+\sum_{t\in \mathrm{N}_2-\{i\}}|a_{it}|x_t\le |a_{ii}|x_i\ (\forall i\in \mathrm{N}_2)$$

Therefore

$$R_i(\boldsymbol{B})=\sum_{j\ne i}|b_{ij}|=\delta\sum_{t\in \mathrm{N}_1}|a_{it}|+\sum_{t\in \mathrm{N}_2-\{i\}}|a_{it}|x_t+\delta\sum_{t\in \mathrm{N}_3}|a_{it}|x_t\le |a_{ii}|x_i=|b_{ii}|$$

For any $i\in \mathrm{N}_3$,since $0<\delta\le 1$ and $\delta\ge \max_{i\in \mathrm{N}_3} M_i$,we obtain

$$\sum_{t\in \mathrm{N}_1}|a_{it}|+\sum_{t\in \mathrm{N}_2}|a_{it}|x_t+\delta\sum_{t\in \mathrm{N}_3-\{i\}}|a_{it}|x_t\le \delta P_i\ (\forall i\in \mathrm{N}_3)$$

Therefore

$$R_i(\boldsymbol{B})=\sum_{j\ne i}|b_{ij}|=\delta\sum_{t\in \mathrm{N}_1}|a_{it}|+\sum_{t\in \mathrm{N}_2}|a_{it}|x_t+\delta\sum_{t\in \mathrm{N}_3-\{i\}}|a_{it}|x_t$$
$$\le \sum_{t\in \mathrm{N}_1}|a_{it}|+\sum_{t\in \mathrm{N}_2}|a_{it}|+\delta\sum_{t\in \mathrm{N}_3-\{i\}}|a_{it}|x_t$$
$$\le \delta P_i=\delta|a_{ii}|=|b_{ii}|$$

To sum up , $|b_{ii}|\ge R_i(\boldsymbol{B})\ (\forall i\in \mathrm{N})$. Therefore B is an irreducible diagonally dominant matrix. Then by Lemma 1, we have that $B\in \overline{\mathrm{D}}$. By Lemma3, we obtain that $A\in \overline{\mathrm{D}}$.

Theorem4. Let, $\boldsymbol{A}=(a_{ij})\in \mathbf{C}^{n\times n}$ **,** $\mathrm{N}_2\setminus \mathrm{J}\ne \phi$ and

$$\mathrm{J}=\{i\,|\,|a_{ii}|x_i=\sum_{t\in \mathrm{N}_2-\{i\}}|a_{it}|x_t+\lambda(\sum_{t\in \mathrm{N}_1}|a_{it}|+\sum_{t\in \mathrm{N}_3}|a_{it}|x_t),\ i\in \mathrm{N}_2\}$$

If $\boldsymbol{A}$ satisfies (8) and for $\forall i\in \mathrm{J}\cup \mathrm{N}_3$, there exists a nonzero elements chain $a_{ir_1}a_{r_1r_2}\cdots a_{r_kk}\ne 0$ such that $k\in \mathrm{N}_2\setminus \mathrm{J}$, then $\boldsymbol{A}\in \overline{\mathrm{D}}$.

Proof. Since, $\forall i\in \mathrm{N}_3$, there exists a nonzero elements chain,we know $\max_{i\in \mathrm{N}_3} M_i>0$ .By the expression of λ ,we know $0<\lambda\le 1$ .So there must exist a positive number δ , such that

$$\lambda\le\delta\le\min\{\frac{1}{\max_{i\in \mathrm{N}_3} x_i},\min_{i\in \mathrm{N}_2} m_i,1\}$$

Constructing positive diagonal matrix $\boldsymbol{D}=\mathrm{diag}\{d_1,d_2,\cdots,d_n\}$, and

$$d_i=\begin{cases}\delta & (i\in \mathrm{N}_1)\\ x_i & (i\in \mathrm{N}_2)\\ \delta x_i & (i\in \mathrm{N}_3)\end{cases}$$

Let matrix $\boldsymbol{B} = \boldsymbol{AD} = (b_{ij})_{n\times n}$.

For any $i \in \mathrm{N}_1$, due to $0 < \delta \le 1$ and $\delta \ge \max_{i\in \mathrm{N}_1} K_i$,we can get

$$\delta \sum_{t\in \mathrm{N}_1-\{i\}} |a_{it}| + \sum_{t\in \mathrm{N}_2} |a_{it}| x_t + \sum_{t\in \mathrm{N}_3} |a_{it}| x_t \le \delta |a_{ii}| \ (\forall i \in \mathrm{N}_1)$$

Therefore

$$R_i(\boldsymbol{B}) = \sum_{j\ne i} |b_{ij}| = \delta \sum_{t\in \mathrm{N}_1-\{i\}} |a_{it}| + \sum_{t\in \mathrm{N}_2} |a_{it}| x_t + \sum_{t\in \mathrm{N}_3} |a_{it}| \delta x_t$$
$$\le \delta \sum_{t\in \mathrm{N}_1-\{i\}} |a_{it}| + \sum_{t\in \mathrm{N}_2} |a_{it}| x_t + \sum_{t\in \mathrm{N}_3} |a_{it}| x_t \le \delta |a_{ii}| = |b_{ii}|$$

For any $i \in \mathrm{N}_2$, since $\delta \le \min_{i\in \mathrm{N}_2} m_i \ (\forall i \in \mathrm{N}_2)$,we can know

$$\delta \sum_{t\in \mathrm{N}_1} |a_{it}| + \delta \sum_{t\in \mathrm{N}_3} |a_{it}| x_t + \sum_{t\in \mathrm{N}_2-\{i\}} |a_{it}| x_t \le |a_{ii}| x_i \ (\forall i \in \mathrm{N}_2)$$

Therefore

$$R_i(\boldsymbol{B}) = \sum_{j\ne i} |b_{ij}| = \delta \sum_{t\in \mathrm{N}_1} |a_{it}| + \sum_{t\in \mathrm{N}_2-\{i\}} |a_{it}| x_t + \delta \sum_{t\in \mathrm{N}_3} |a_{it}| x_t \le |a_{ii}| x_i = |b_{ii}|$$

For any $i \in \mathrm{N}_3$, since $0 < \delta \le 1$ and $\delta \ge \max_{i\in \mathrm{N}_3} M_i$,we obtain

$$\sum_{t\in \mathrm{N}_1} |a_{it}| + \sum_{t\in \mathrm{N}_2} |a_{it}| x_t + \delta \sum_{t\in \mathrm{N}_3-\{i\}} |a_{it}| x_t \le \delta P_i \ (\forall i \in \mathrm{N}_3)$$

Therefore

$$R_i(\boldsymbol{B}) = \sum_{j\ne i} |b_{ij}| = \delta \sum_{t\in \mathrm{N}_1} |a_{it}| + \sum_{t\in \mathrm{N}_2} |a_{it}| x_t + \delta \sum_{t\in \mathrm{N}_3-\{i\}} |a_{it}| x_t$$
$$\le \sum_{t\in \mathrm{N}_1} |a_{it}| + \sum_{t\in \mathrm{N}_2} |a_{it}| + \delta \sum_{t\in \mathrm{N}_3-\{i\}} |a_{it}| x_t$$
$$\le \delta P_i = \delta |a_{ii}| = |b_{ii}|$$

To sum up ,$|b_{ii}| \ge R_i(\boldsymbol{B}) \ (\forall i \in \mathrm{N})$. Therefore B is a diagonally dominant matrix with nonzero elements chain. Then by Lemma 2, we have that $B \in \overline{\mathrm{D}}$. By Lemma3, we obtain that $A \in \overline{\mathrm{D}}$.

2. EXAMPLE

Example1. Let

$$\boldsymbol{A} = \begin{pmatrix} 3 & 3 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 4 \end{pmatrix}$$

Then $\mathrm{N}_1 = \{1\}$, $\mathrm{N}_2 = \{2\}$, $\mathrm{N}_3 = \{3\}$. We can obtain $r = 1/4$, $P_3 = 1$, $x_2 = 1/3$, $x_3 = 1/4$, $M_3 = 1/3$, $K_1 = 1/3$, $\lambda = \max\{M_3, K_1\} = 1/3$. Since

$$|a_{22}| x_2 = 2/3 > \lambda(|a_{21}| + |a_{23}| x_3) = 1/2$$

Therefore $\boldsymbol{A}$ satisfies the condition of theorem 2 in this paper, so $\boldsymbol{A} \in \overline{\mathrm{D}}$. But

$$|a_{22}| x_2 = 2/3 < |a_{21}| + |a_{23}| x_3 = 3/2$$

So $\boldsymbol{A}$ does not satisfy the corresponding conditions of the Theorem in paper [1].

Take positive diagonal matrix $D = \mathrm{diag}\{7/18, 1/3, 1/9\}$,so

$$AD=\begin{pmatrix} 7/6 & 1 & 0 \\ 7/18 & 2/3 & 2/9 \\ 0 & 1/3 & 4/9 \end{pmatrix}$$

That is $\boldsymbol{A}\in \overline{\mathrm{D}}$.

Example2. Let

$$A=\begin{pmatrix} 3 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2.5 \\ 0.1 & 1 & 4 & 3.9 \\ 0.5 & 0.5 & 0.5 & 5 \end{pmatrix}$$

Then $\mathrm{N}_1=\{1\}$, $\mathrm{N}_2=\{2,3\}$, $\mathrm{N}_3=\{4\}$. We can obtain $r=0.3$, $P_4=1.5$, $x_2=0.2$, $x_3=0.2$, $x_4=0.3$, $M_4=7/15$, $K_1=7/30$, $\lambda=\max\{M_4,K_1\}=7/15$. Since

$$|a_{22}|x_2=0.4>|a_{23}|x_3+\lambda(|a_{21}|+|a_{24}|x_4)=0.35$$

$$|a_{33}|x_3=0.8>|a_{32}|x_2+\lambda(|a_{31}|+|a_{34}|x_4)=0.79267$$

Therefore $\boldsymbol{A}$ satisfies the condition of theorem 2 in this paper, so $\boldsymbol{A}\in \overline{\mathrm{D}}$. But

$$|a_{22}|x_2=0.4<|a_{21}|+|a_{23}|x_3+|a_{24}|x_4=0.75$$

So $\boldsymbol{A}$ does not satisfy the corresponding conditions of the Theorem in paper [3].

Take positive diagonal matrix $D=\mathrm{diag}\{0.47,0.2,0.2,0.14\}$,so

$$\mathbf{AD}=\begin{pmatrix} 1.41 & 0.2 & 0.2 & 0.14 \\ 0 & 0.4 & 0 & 0.35 \\ 0.047 & 0.2 & 0.8 & 0.546 \\ 0.235 & 0.1 & 0.1 & 0.7 \end{pmatrix}$$

That is $\mathbf{A}\in \overline{\mathrm{D}}$.

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REFERENCES

[1] Gan Taibin, Huang Tingzhu.Simple Criteria for Nonsingular H-matrices. Liner AlgebraAppl.,374 (**2003**):317-326

[2] T.Z.Huang. A note on generalized diagonally dominant matrices.*Liner Algebra Appl.*, 225(**1995**): 237-242.

[3] Tuo Qing, Zhu Li, Liu Jianzhou. One type of new criteria conditions for Nonsingular H-matrices. *Computing Mathematics*, 2(**2008**)177-182, in Chinese.

[4] Rafael Bru, Isabel Giménez, Apostolos Hadjidimos.Is $A\in C_{n\times n}$ a general H-matrix? *Liner Algebra Appl.*, 436(2) (**2012**):364-380.

[5] M. Alanelli , A. Hadjidimos. A new iterative criterion for H-matrices:The reducible case. *Liner Algebra Appl.*, 428(**2008**):2761-2777.

[6] F.O.Farid. Notes on matrices with diagonally dominant properties. *Liner Algebra Appl.*, 435(**2011**):2793-2812.

[7] Toshiyuki Kohno, Hiroshi Niki, Hideo Sawami, Yi-ming GAO. An iterative test for H-matrix. *Journal of Computational and Applied Mathematics*.115 (**2000**)349-355.

[8] Chi –Ye Wu,Ting-Zhu Huang. Stability of block LU factorization for block tridiagonal blocks H-matrices. *Journal of Computional and Mathematics*.236(**2012**):2673-2684

[9] Ping-Fan Dai. Error bounds for linear complementarity problems of DB-matrices. *Liner Algebra Appl.*,434 (**2011**):830-840

[10] R.S. Varga. On recurring theorems on diagonal dominance. *Linear Algebra Appl.* , 13(**1976**)1-9

[11] P.N. Shivakumar, K.H. Chew. A sufficient condition for no vanishing of determinants. *Proc. Amer. Math. Soc*., 43(**1974**)63-66
[12] T.Z. Huang. Practical sufficient conditions of nonsingular H-matrices. *Numer. Math. Sinica*, 3 (**1993**) 318-328.
[13] A, Berman, R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. SIAM Press. Philadelphia,**1994**
[14] Huang ZeJun, Liu JianZhou. New Iterative Criteria for Nonsingular H-matrix .*Journal of Engineering Mathematics*, **2008**, 25(5): 939-942. in Chinese
[15] W.Li. On Nekrasov matrices. *Liner Algebra Appl*., 281(**1998**):87-96.
[16] Ángel Cuenca, Ricardo Pizá, Julián Salt, and Antonio Sala. Linear Matrix Inequalities in Multirate Control over Networks. *Mathematical Problems in Engineering*, Volume **2012** (**2012**), Article ID 768212, 22 pages.
[17] E.Momoniat. Matrix Exponentiation and the Frank-Kamenetskii Equation. *Mathematical Problems in Engineering*, Volume 2012 (**2012**), Article ID 713798, 14 pages.