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Some conclusions of the exchangeable random variables and the independent identical distribution random variables

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ABSTRACT

As the fundamental structure theorem of infinite exchangeable random variables sequences, the Definetti's theorem does not work to finite exchangeable random variables sequences, it is therefore necessary to find other techniques to solve the approximate behavior problems of finite exchangeable random variables sequences. By using reverse martingale approach, some scholars have given some results. In this paper we do some researches about the similarity and difference of identically distributed random variables and exchangeable random variables sequences, mainly discuss the limit theory of exchangeable random variables.

Key words: Exchangeable random variable; Exchangeability; Law of large numbers

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INTRODUCTION

Limit theory mainly study independent random variables, but in many practical problems, samples are not independent, or the function of independent sample is not independent, or the verification of independence is more difficult. So the concept of dependent random variables in probability and statistics is mentioned. Exchangeable random variables are major type of dependent random variables. If the replacement the joint distribution of $X_1, X_2, \cdots, X_n$ is unchanged, that is, for each replacement of $1, 2, \cdots, n$ the joint distribution of $X_1, X_2, \cdots, X_n$ is the same with that of $X_{\pi(1)}, X_{\pi(2)}, \cdots, X_{\pi(n)}$; then the random variable finite series $X_1, X_2, \cdots, X_n$ is known as the exchangeable. Obviously, the independent identical distribution random variables are the simplest exchangeable random variables. The concept of exchangeable random variables is the first proposed by De Finetti 1930.The most famous property of exchangeable random variables is its basic structured theorem, called De Finetti theorem; that is, the infinite series of exchangeable random variables is independent identical distribution, if its tail is σ algebra. Some scholars have given some results about exchangeable random variables sequences ([1]-[5]).The aim of this paper is generalize the independent identical distribution variables [6] and[7] to the exchangeable random variables. As the selection method for truncated random variables is different when deal with random variables, so the prove method is more simple than that of [6] and [7].

THEDEFINITION AND THE LEMMA

Definition [8]. The positive valued function $l(x)$ defined on $[0,\infty)$ is called slowly changed, if for any $c>0$,we have $\lim\limits_{x\to\infty}\frac{l(cx)}{l(x)}=1$ Suppose $\{a_{ni}, 1\le i\le n, n\ge 1\}$ is real positive series that satisfy $A^{\alpha}_{\alpha,n}=n^{-1}\sum_{i=1}^{n}|a_{ni}|^{\alpha}$ and

$$A^{\alpha}_{\alpha,n}=n^{-1}\sum_{i=1}^{n}|a_{ni}|^{\alpha} \tag{1}$$

Lemma [8]. Suppose $\{X_n, n\ge 1\}$ are exchangeable random variables, which satisfy $Cov\left(f_1(X_1), f_2(X_2)\right)\le 0$

Let $A_1, A_2, \cdots A_m$ be the disjoint non-empty subset of $\{1, 2, \cdots, n\}$ with $m \ge 2$, suppose $f_i, i = 1, 2, \cdots, m$ is a non-increase

(Non-decrease) function, then

(1)If $f_i \ge 0, i = 1, 2, \cdots, m$ then $E\left(\prod_{i=1}^{n} f_i\left(X_j, j \in A_i\right)\right) \le \prod_{i=1}^{n} Ef_i\left(X_j, j \in A_i\right)$

(2)Particularly, for any $x_i \in R, i = 1, 2, \cdots, m,$ we have $P\left(X_1 < x_1, \cdots X_m < x_m\right) \le \prod_{i=1}^{n} P\left(X_i < x_i\right)$

Subsequently, we will outline several lemmas, which will be used in the proof of the main theorems. If necessary, we will also give the proof.

Lemma2 Suppose $X_1, X_2, \cdots, X_n$ are exchangeable random variables, that satisfy $Cov\left(f_1\left(X_1\right), f_2\left(X_2\right)\right) \le 0$

$EX_k = 0, \sigma_k^2 = EX_k^2 < \infty\left(k = 1, 2, \cdots, n\right)$, Suppose there exists a positive constant H such that

$\left|EX_k^m\right| \le \frac{m!}{2} \cdot \sigma_k^2 H^{m-2}\left(k = 1, 2, \cdots, n\right)$

then we have $P\left(\sum_{i=1}^{n} X_i \ge x\right) \le \exp\left(-x^2 / 4\sum_{i=1}^{n}\sigma_i^2\right),\ 0 \le x \le \sum_{i=1}^{n}\sigma_i^2 / H$

$$P\left(\sum_{i=1}^{n} X_i \le -x\right) \le \exp\left(-x^2 / 4\sum_{i=1}^{n}\sigma_i^2\right),\ 0 \le x \le \sum_{i=1}^{n}\sigma_i^2 / H$$

Proof. Based on Theorem 2.5 in[9] and Lemma 1 in [7], this Lemma is easy to prove.

Lemma 3. Suppose $\{X, X_n; n \ge 1\}$ are the exchangeable random variables and there exist

$h > 0, r > 0$, such that $E\left[\exp\left(h\left(x\right)^r\right)\right] < \infty$ (2) $\left\{X_{ni}, 1 \le i \le n, n \ge 1\right\}$ are the exchangeable random variables that satisfy $Cov\left(f_1\left(X_{n1}\right), f_2\left(X_{n2}\right)\right) \le 0$

$EX_{ni} = 0,\ \left\{a_{ni}, 1 \le i \le n, n \ge 1\right\}$ are real constant array that satisfy

(i) There exist $\beta, 0 < \beta \le r$ with $\lim_{n\to\infty} u_n = 0$ and $\left\{u_n, n \ge 1\right\}$, such that $\left|a_{ni}X_{ni}\right| \le \frac{u_n\left|X_i\right|^\beta}{\log n}$ *a.s.*

(ii) There exists $\delta > 0$ and array $\left\{v_n, n \ge 1\right\}$, that satisfy $\lim_{n\to\infty} v_n = 0$, $X_{ni}^2\sum_{i=1}^{n} a_{ni}^2 \le \frac{v_n\left|X_i\right|^\delta}{\log n}$ *a.s.*

Proof. Based on Theorem 2.5 in [9]and Theorem 18 in[10], the lemma is easy to prove

Lemma 4. Suppose $\{X, X_n; n \ge 1\}$ are the exchangeable random variables and there exist

$h > 0,\ r > 0$, such that $E\left[\exp\left(h\left(x\right)^r\right)\right] < \infty$

$\left\{X_{ni}, 1 \le i \le n, n \ge 1\right\}$ are the exchangeable random variables that satisfy

$Cov\left(f_1\left(X_{n1}\right), f_2\left(X_{n2}\right)\right) \le 0$

$EX_{ni} = 0,\ 1 \le i \le n, n \ge 1,\ \left\{a_{ni}, 1 \le i \le n, n \ge 1\right\}$ are real constant array that satisfy

(1) $E\left[\exp\left(h\left(x\right)^r\right)\right] < \infty$

(2) $\beta, 0 < \beta \le r$ and constant c>0 ,such that $\left|a_{ni}X_{ni}\right| \le \frac{c\left|X_i\right|^\beta}{\log n}$ *a.s.*

(3) There exists $\delta > 0$ and array $\{\upsilon_n, n \ge 1\}$, $\lim_{n\to\infty} v_n = 0$, such that $X_{ni}^2 \sum_{i=1}^n a_{ni}^2 \le \frac{v_n |X_i|^\delta}{\log n}$ $a.s.$

then $\sum_{i=1}^n a_{ni} X_i \to 0$ $a.s.$ $n \to \infty$

THEMAIN RESULTS AND PROOF

Theorem 1. Suppose $\{X, X_n; n \ge 1\}$ are the exchangeable random variables that satisfy $Cov\left(f_1(X_1), f_2(X_2)\right) \le 0$ Suppose f_i, $i = 1, 2$ are functions satisfy the above rule and non-decrease with $X_1, X_2, EX_1 = 0$, $\alpha p > 1$, $l(x) > 0$ is monotonous non-decrease function when $x \to +\infty$, $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ are real constant array with,

$A_{\alpha,n}^\alpha = n^{-1} \sum_{i=1}^n |a_{ni}|^\alpha$, Further, suppose $A_\alpha = \lim_{n\to\infty} \sup A_\alpha, n < \infty$, $E|X|^\beta < \infty$, $EX = 0$

and $1 < \alpha, \beta < \infty$, $1 < p < 2$, 且 $\frac{1}{p} = \frac{1}{\alpha} + \frac{1}{\beta}$, then

$$\frac{1}{n^{1/p}} \sum_{i=1}^n a_{ni} X_i \to \infty \ a.s. \ (n \to \infty) \tag{3}$$

Proof. Without loss of generality, for any $1 \le i \le n, n \ge 1$, suppose $a_{ni} > 0$, as $\{X, X_n; n \ge 1\}$ are the exchangeable random variables, and $a_{n1}X_1, a_{n2}X_2, \cdots, a_{nn}X_n$ also satisfy

$$Cov\left(f_1(a_{n1}X_1), f_2(a_{n2}X_2)\right) \le 0$$

and $\frac{1}{p} = \frac{1}{\alpha} + \frac{1}{\beta}$, Then $p < \alpha \wedge \beta \wedge 2$. from (1) we have

$$E\left|n^{-1/p} \sum_{i=1}^n a_{ni} X_i\right|^{\alpha\wedge\beta\wedge 2} \le Cn^{-\alpha\wedge\beta\wedge 2/p} \sum_{i=1}^n |a_{ni}|^{\alpha\wedge\beta\wedge 2} E|X|^{\alpha\wedge\beta\wedge 2} \le Cn^{-\alpha\wedge\beta\wedge 2/p+1} A_{\alpha\wedge\beta\wedge 2,n}^{\alpha\wedge\beta\wedge 2} \to 0, n \to \infty$$

then $n^{-1/p} \sum_{i=1}^n a_{ni} X_i \xrightarrow{p} 0$ $n \to \infty$

From the symmetrized inequality proved in Lemma 14 in [10], we know that, in order to prove $n^{-1/p} \sum_{i=1}^n a_{ni} X_i \xrightarrow{p} 0, n \to \infty$, we just need to prove

$$\frac{1}{n^{1/p}} a_{ni} X_i^S \to 0 \quad a.s. \ n \to \infty$$

where X_i^S is the symmetrized form of X_n, From Lemma 3 in [11], we have the symmetrized series of $Cov\left(f_1(X_1), f_2(X_2)\right) \le 0$

also satisfy the inequality, i.e. $Cov\left(f_1(X_1^S), f_2(X_2^S)\right) \le 0$, Without loss of generality, we assume that $\{X_n, n \ge 1\}$ are the symmetrized exchangeable random variables that satisfy $Cov\left(f_1(X_1), f_2(X_2)\right) \le 0$ for all $1 \le i \le n$, $n \ge 1$, Letting

$$X_i' = X_i I\left(|X_i| \le n^{1/\beta}\right) + n^{1/\beta} I\left(X_i > n^{1/\beta}\right) - n^{1/\beta} I\left(X_i < -n^{1/\beta}\right)$$

$$X_i'' = X_i I\left(|X_i| > n^{1/\beta}\right) - n^{1/\beta} I\left(X_i > n^{1/\beta}\right) + n^{1/\beta} I\left(X_i < -n^{1/\beta}\right)$$

$$\bar{X}_i'' = X_i I\left(|X_i| > n^{1/\beta}\right)$$

$a'_{ni} = a_{ni} I\left(\left|a_{ni}\right| \le n^{1/\alpha}\right)$

$a''_{ni} = a_{ni} - a'_{ni} = a_{ni} I\left(\left|a_{ni}\right| > n^{1/\alpha}\right)$

then

$$\sum_{i=1}^{n} a_{ni} X_i = \sum_{i=1}^{n} a'_{ni} X'_i + \sum_{i=1}^{n} a''_{ni} X'_i + \sum_{i=1}^{n} a_{ni} X''_i \qquad (4)$$

As $\frac{1}{p} = \frac{1}{\alpha} + \frac{1}{\beta}, \beta = \frac{\alpha}{\alpha - 1}\left\{1 + \beta\left(1 - \frac{1}{p}\right)\right\}$, $\left|\bar{X}''_i\right| \le \left|\bar{X}''_i\right|^{\beta(\alpha-1)/\alpha} n^{-(1-1/p)}$

and $E\left|X\right|^{\beta} < \infty$ which is equivalent to $\sum_{n=1}^{\infty} P\left(\left|X\right|^{\beta} > n\right) < \infty$,then $\sum_{n=1}^{\infty} P\left(\left|X_n\right|^{\beta} > n\right) < \infty$,

From Borel-Cantelli Lemma, we have $P\left(\left|X_n\right|^{\beta} > n, i.o.\right) = 0$。 hence

$$\frac{1}{n}\sum_{i=1}^{n}\left|\bar{X}''_i\right|^{\beta} \to 0 \ a.s. \left(n \to \infty\right)$$

From *Hölder* inequality, $\left|X''_i\right| \le \left|\bar{X}''_i\right|$ 及 $\left|\bar{X}''_i\right| \le \left|\bar{X}''_i\right|^{\beta(\alpha-1)/\alpha} \cdot n^{-(1-1/p)}$ and

$$\frac{1}{n}\sum_{i=1}^{n}\left|\bar{X}''_i\right|^{\beta} \to 0 \ a.s. \left(n \to \infty\right)$$

then

$$n^{-1/p}\left|\sum_{i=1}^{n} a_{ni} X''_i\right| \le n^{-1/p}\sum_{i=1}^{n}\left|a_{ni}\right|\left|\bar{X}''_i\right| \le n^{-1}\sum_{i=1}^{n}\left|a_{ni}\right|\left|X''_i\right|^{\beta(\alpha-1)/\alpha}$$

$$\le A_{\alpha,n}\left(\frac{1}{n}\sum_{i=1}^{n}\left|X''_i\right|^{\beta}\right)^{(\alpha-1)/\alpha} \to 0 \ a.s. \left(n \to \infty\right)$$

Therefore, we have

$$n^{-1/p}\sum_{i=1}^{n} a_{ni} X''_i \to 0 \ a.s. \left(n \to \infty\right) \qquad (5)$$

As $\frac{1}{p} = \frac{1}{\alpha} + \frac{1}{\beta}$, $\alpha \vee \beta < 1$,then

$$1 + \frac{(2-\alpha)^{+}}{\alpha} + \frac{(2-\beta)^{+}}{\beta} = \begin{cases} \frac{2}{\alpha \wedge \beta} & if\, \alpha \wedge \beta < 1 \\ 1 & if\, \alpha \wedge \beta \ge 1 \end{cases}$$

Therefore, we have $\sum_{i=1}^{n} E\left(a'_{ni} X'_i\right)^2 \le CnA_{\alpha\wedge 2,n}^{\alpha\wedge 2} \cdot n^{(2-\alpha)^{+}/\alpha + (2-\beta)^{+}/\beta}\left\|X\right\|_{\beta\wedge 2}^{\beta\wedge 2} = O\left(\max\left\{n^{2/\alpha}, n^{2/\beta}, n\right\}\right)$

Moreover, for any $1 \le i \le n, n \ge 1$,we have $\left|n^{-1/p} a'_{ni} X'_i\right| \le n^{1/\alpha} n^{1/\beta} n^{-1/p} = 1$,and

$\max\left\{n^{2/\alpha}, n^{2/\beta}, n\right\} = O\left(n^{2/p}\log^{-2} n\right)$, From

Lemma 2, for sucient small " and sucient large n, we have

$$P\left(n^{1/p}\sum_{i=1}^{n} a'_{ni} X'_i > \varepsilon\right) \le \exp\left(\frac{-\varepsilon^2}{4n^{-2/p} O\left(\max\left\{n^{2/\alpha}, n^{2/\beta}, n\right\}\right)}\right) \le \exp\left(-\varepsilon^2\left(\log n\right)^2\right)$$

By the same procedures, we can also prove that $P\left(n^{-1/p}\sum_{i=1}^{n}a'_{ni}X'_i<-\varepsilon\right)\le\exp\left(-\varepsilon^2\left(\log n\right)^2\right)$,

So $\sum_{i=1}^{n}P\left(n^{-1/p}\sum_{i=1}^{n}a'_{ni}X'_i>\varepsilon\right)<\infty$。 then

$$n^{-1/p}\sum_{i=1}^{n}a'_{ni}X'_i\to 0\ a.s.\left(n\to\infty\right) \tag{6}$$

For $\frac{1}{p}=\frac{1}{\alpha}+\frac{1}{\beta}$, we have $n^{-1/p}\left|\sum_{i=1}^{n}a''_{ni}X'_i\right|\le n^{-1/p}n^{1/\beta}\sum_{i=1}^{n}\left|a_{ni}\right|I\left(a_{|ni|}>n^{1/\alpha}\right)$

$$\le n^{-1/p+1/\beta}n^{(1-\alpha)/\alpha}\sum_{i=1}^{n}\left|a_{ni}\right|^{\alpha}=A^{\alpha}_{\alpha,n} \tag{7}$$

Then from (4),(5),(6),(7), we have

$$\limsup_{n\to\infty}n^{-1/p}\left|\sum_{i=1}^{n}a_{ni}X_i\right|\le A^{\alpha}_{\alpha}\ a.s.\left(n\to\infty\right)$$

By replacing X_i with tX_i we have

$$\limsup_{n\to\infty}n^{-1/p}\left|\sum_{i=1}^{n}a_{ni}X_i\right|\le \frac{A^{\alpha}_{\alpha}}{t}\ a.s.\left(n\to\infty\right)$$

Let $t\to\infty$,we have

$$\frac{1}{n^{1/p}}\sum_{i=1}^{n}a_{ni}X_i\to\infty\ a.s.\left(n\to\infty\right)$$

the inequality (3) is true.

CONCLUSION

As the fundamental structure theorem of infinite exchangeable random variables sequences, the Definetti's theorem states that infinite exchangeable random variables sequences is independent and identically distributed with the condition of the tail σ-algebra. So some results about independent identically distributed random variables is similar to exchangeable random variables. As the fundamental structure theorem of infinite exchangeable random variables sequences, the Definetti's theorem does not work to finite exchangeable random variables sequences, it is therefore necessary to find other techniques to solve the approximate behavior problems of finite exchangeable random variables sequences. By using reverse martingale approach, some scholars have given some results. In this paper we do some researches about the similarity and difference of identically distributed random variables and exchangeable random variables sequences, mainly discuss the limit theory of exchangeable random variables.

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