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Research on generalized space-time fractional convection-diffusion equation

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ABSTRACT

By studying the basic theory of fractional differential equation, the paper extends the space-time fractional convection-diffusion equation. We use $\gamma(0<\gamma\le 1)$ order fractional derivative instead of the space first derivative in the space-time fractional convection-diffusion equation and then obtain generalized space-time fractional convection-diffusion equation. We discuss its numerical solution and construct implicit difference scheme and then prove the stability and convergence of the difference scheme. Next we use the variation iteration method to discuss its approximate analytical solution and we give numerical examples to verify the effectiveness of method.

Key words: Krylov subspace, PCGMRES algorithm, space-time fractional convection-diffusion equation, fractional differential equation

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INTRODUCTION

For the most practical engineering problems, the core of the engineering calculation is computation of fractional calculus, for example in the field of electrochemical process、colored noise、control theory、hydromechanics、chaos and biological engineering. This is mainly due to fractional calculus with describing the characteristic of substance memory function and genetic effect. This characteristic makes it more accurate than the description of the integer-order derivative. For example, Chen Bingsan and Huang Yijian[1] introduce fractional calculus into the constitutive equation of viscoelastic fluid.

The basic theory of space fractional differential equation is proposed by Feller[2] in 1952. Now many scholars work on space fractional differential equation. Mainardi et al[3] considered the space fractional diffusion equation and give its explicit representation of the Green's function. In 1989, Schneider and Wyss[4] studied time fractional diffusion wave equation and they studied the property of the corresponding Green's function. In 2000, Benson et al[5] considered the space fractional reaction-diffusion equation and obtained the analytical solution. In 2003, Liu et al[6] used variable substitution、Mellin transform、Laplace transform and H function property to get the complete solution of time fractional advection-diffusion equation. In terms of numerical solution, Liu et al[7] proposed fractional line method and used the method to transform fractional partial differential equation for ordinary differential system space fractional diffusion equation and they used it to simulate groundwater transport. In the same year, Lin Ran and Liu Fawang[8] considered the simplest fractional ordinary differential equation and introduced fractional linear multi-step method. Lu Xuanzhu[9] studied the numerical solution of time fractional constant coefficient convection-diffusion equation and proposed a numerical calculation method only needs to store part of historical data. In 2005, Hu Yizheng et al[9] studied the four-term differential equation of fractional power control system and proved existence and uniqueness of the solution and then proposed three numerical solution methods to simulate analytical solution. Finally they gave numerical examples to show that the three proposed numerical methods can be used to simulate the behavior of fractional control system. In 2006, Chen Chunhua and Lu Xuanzhu[1] studied the numerical solution of the initial boundary value problem of fractional diffusion equation and obtained an unconditionally stable and conditionally convergent implicit finite difference scheme. In the same year, Wang Xuebin[2] considered multinomial fractional ordinary differential equation. He proved existence and uniqueness of the solution and proposed three numerical solution methods to approximate the equation solution.

Using fractional derivative model, it will lead to a series of fractional differential equations in most cases. For these equations, people have found several classes solving method. Although analytical solution of some equations can seek out, but many analytic solutions for fractional differential equation are made more special function to represent and it is difficult to express numerically these special functions. Moreover, some nonlinear equations can not find its analytical solution. So, numerical method in the computation of fractional differential equation is more important. However, with respect to the integer-order differential equation, development of numerical method for solving fractional differential equations is still quite immature. It is known that some existing numerical methods can not be applied into nonlinear fractional differential equation. Currently there are many of these methods lack of system stability and convergence analysis. So this paper deeply studies stability、convergence and approximate analytical solution of space-time fractional convection-diffusion equation.

GENERALIZED SPACE - TIME FRACTIONAL ADVECTION-DIFFUSION EQUATION

Basic theory of fractional differential equation

There are three fractional derivative definitions: Griinwald-letnikov (G-L) Definition、Riemann-Liouville (R-L) Definition and Caputo Definition. R-L Definition:

Let $\gamma \in (0,1), a, b \in R, a < t < b, f(t)$ is continuous on $[a,b]$, the R-L fractional differential is ${}_a\mathrm{D}_t^{1-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \frac{d}{dt} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\gamma}} d\tau$

where $\Gamma(\cdot)$ is Gamma function. Caputo Definition: ${}_a D_t^{\gamma} f(t) = \frac{1}{\Gamma(1-\gamma)} \int_a^t f(s)(t-s)^{-\gamma} ds, 0 \le t \le T, 0 < \gamma < 1.$

When γ negative real number or positive integer, three definitions is can be converted to each other. G-L definition is generally used for discrete computing. R-L and Caputo definition are commonly used in the discussion of fractional differential equations.

Space - Time fractional advection-diffusion equation

In the paper, we use $\gamma (0 < \gamma \le 1)$ order fractional derivative instead of the space first derivative in the space-time fractional convection-diffusion equation and then obtain generalized space-time fractional convection-diffusion equation.

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = a(x) \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} - b(x) \frac{\partial^{y} u(x,t)}{\partial x^{y}} + q(x,t), 0 < \alpha \le L, 0 < x \le T \tag{1}$$

Where $a(x) > 0, b(x) > 0, 0 < \alpha, \gamma \le 1, 1 \le \beta < 2$, $u(x,0) = f(x), 0 < x < L$, $u(0,t) = u(L, t = 0), t > 0$, $\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}}$ is Caputo fractional derivative, $\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}}$ and $\frac{\partial^{\gamma} u(x,t)}{\partial x^{\gamma}}$ are Riemann-Liouville fractional derivative.

NUMERICAL SOLUTION OF GENERALIZED SPACE - TIME FRACTIONAL ADVECTION-DIFFUSION EQUATION

Lemma 3.1 If $f \in L_1(R)$ and $f \in C^{\alpha+1}(R)$, then $A_h f(x) \in Af(x) + o(h)$, where $A_h f(x) \in \frac{1}{\Gamma(-\alpha)} \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} f(x-(k-1)h), Af(x) = \frac{d^{\alpha} f(x)}{dx^{\alpha}}$, h is step and $\frac{d^{\alpha} f(x)}{dx^{\alpha}}$ is Riemann-Liouville fractional derivative.

Difference scheme

Given equidistant division: $t_k = k\tau, k = 0,1,2\cdots,n, x_i = ih, i = 1,2\cdots,m$, where $\tau = \frac{T}{n}$ and $h = \frac{L}{m}$ respectively denote time step and space step. Time fractional derivative can be approximated as follows

$$\frac{\partial^\alpha u(x_i,t_{k+1})}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{u(x_i,t_{j+1})-u(x_i,t_j)}{\tau}\int_{j\tau}^{(j+1)\tau}\frac{d\xi}{(t_{k+1}-\xi)^\alpha}+o(\tau)$$
$$\approx\frac{\tau^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{u(x_i,t_{t-j+1})-u(x_i,t_{k-j})}{\tau}[(j+1^{1-\alpha}-j^{1-\alpha}) \qquad (2)$$

For two space fractional derivative, we use Grunwald modified formula to replace (from lemma 3.1 we have that fractional derivative and Grunwald modified discrete formula are uniformly convergent):

$$\frac{\tau^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\sum_{j=1}^{k}\frac{u(x_i,t_{k+1-j})-u(x_i,t_{k-j})}{\tau}[(j+1)^{1-\alpha}-j^{1-\alpha}]+o(\tau)=$$
$$\frac{a(x_i)}{h^\beta}\sum_{j=0}^{i+1}g_j u(x_i-(j-1)h,t_{k+1})+\frac{b(x_i)}{h^\gamma}\sum_{j=0}^{i+1}v_j u(x_i-(j-1)h,t_{k+1})+o(\tau+h)+q(x_i,t_{k+1}) \qquad (3)$$

Let u_i^k be approximate solution of (1), we simplify (3) to implicit difference scheme:
When $k=0$,

$$-(r_i-p_i)u_{i+1}^1+(1+v_1p_i-g_1r_i)u_i^1-(r_ig_2-p_iv_2)u_{i-1}^1-\sum_{j=3}^{i+1}(r_ig_j-p_iv_j)u_{i-j+1}^1u_i^0+\tau^\alpha\Gamma(2-\alpha)q_i^1=\omega q_i^1 \qquad (4)$$

When $k>0$,

$$-(r_i-p_i)u_{i+1}^{k+1}+(1+v_1p_i-g_1r_i)u_i^{k+}-(r_ig_2-p_iv_2)u_{i-1}^{k+1}-\sum_{j=3}^{i+1}(r_ig_j-p_iv_j)u_{i-j+1}^{k+1}$$
$$=u_i^k-\sum_{j=1}^{k}(u_i^{k+1-j}-u_i^{k-j})c_j+\tau^\alpha\Gamma(2-\alpha)q_i^{k+1}$$
$$=(2-2^{1-\alpha})u_i^k+\sum_{j=1}^{k-1}u_i^{k-j}[2(j+1)^{1-\alpha}-(j+2)^{1-\alpha}-j^{1-\alpha}]+c_ku_i^0+\tau^\alpha\Gamma(2-\alpha)q_i^{k+1}$$
$$=\sum_{j=1}^{k-1}d_{j+1}u_i^{k-j}+c_ku_i^0+\omega q_i^{k+1} \qquad (5)$$

Let (3), (4) be written in matrix form

$$\begin{cases}AU^1=U^0+\omega Q^1,\\ AU^{k+1}=d_1U^k+d_2U^{k-1}+\cdots d_kU^1+c_kU^0+\omega q^{k+1},\\ U^0=f.\end{cases} \qquad (6)$$

From $p_i>r_i$ and $1+r_i(1+\beta)>p_i(1+\gamma)$, (5) has the unique solution.

Stability Analysis

Theorem 3.2 Implicit difference scheme approximation defined by (4) and (5) is unconditionally stable.

Proof: Assume $\tilde{u}_i^k, u_i^k$ respectively are the solution of (4) and (5), the computation of q_i^k is accurate. Then the error $\varepsilon_i^k=\tilde{u}_i^k-u_i^k$ satisfies
When $k=0$,

$$-(r_i-p_i)\varepsilon_{i+1}^1+(1+v_1p_i-g_1r_i)\varepsilon_i^1-(r_ig_2-p_iv_2)\varepsilon_{i-1}^1-\sum_{j=3}^{i+1}(r_ig_j-p_iv_j)\varepsilon_{i-j+1}^1=\varepsilon_i^0;$$

When $k>0$,

$$-(r_i - p_i)\varepsilon_{i+1}^{k+1} + (1 + v_1 p_i - g_1 r_i)\varepsilon_i^{k+1} - (r_i g_2 - p_i v_2)\varepsilon_{i-1}^1 - \sum_{j=3}^{i+1}(r_i g_j - p_i v_j)\varepsilon_{i-j+1}^{k+1}\sum_{j=0}^{k-1} d_{j+1}\varepsilon_i^{k-j} + c_k \varepsilon_i^0.$$

The above two equations can be written in matrix form

$$\begin{cases} AE^1 = E^0, \\ AE^{k+1} = d_1 E^k + d_2 E^{k-1} + \cdots + d_{k-1}E^2 + c_k E^0 \end{cases}$$

where $E^k = [\varepsilon_1^k, \varepsilon_2^k, \cdots \varepsilon_{m-1}^k]$. By mathematical induction, we prove $\|E^k\|_\infty \le M\|E^0\|_\infty$. So the implicit difference scheme approximation defined by (4) and (5) is unconditionally stable.

Convergence Analysis

Let $u(x_i, t_k)$ is the exact solution of differential equation in the grid point (x_i, t_k), $e_i^k = u(x_i, t_k) - u_i^k$ and $e_i^k = (e_1^k, e_2^k \cdots, e_{m-1}^k)'$. We have $u_i^k = u(x_i, t_k) - e_i^k$. Next we put it into difference scheme. Using $e^0 = 0$ 、Taylor theorem and integral mean value theorem, we get

$$\frac{\tau^{-\alpha}}{\Gamma\ (2-\alpha)}\sum_{j=0}^{k-1} c_j [u(x_i, t_{t_{k+1-j}}) - u(x_i, t_{t_{k-j}})] = \frac{\tau^{-\alpha}}{\Gamma\ (2-\alpha)}\sum_{j=0}^{k-1} c_j [\frac{\partial u(x_i, t_{t_{k+1}} - j\tau)}{\partial r} + M(\tau)].$$

Because $\frac{\partial u(x_i, t_{k+1} - j\tau)}{\partial r} - \frac{\partial u(x_i, t_{k+1} - \zeta_j)}{\partial r} = M(\tau)$, so we get $\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum c_j[u(x_i, t_{k+1-j}) - u(x_i, t_{k-j})] = \frac{\partial^\gamma u(x_i, t_{k+1})}{\partial x^\gamma} + m_3(\tau + h)$.

By mathematical induction, we prove $\|e^k\|_\infty \le c_s^{-1} m(\tau^{1+a} + h^a)$. So the difference scheme is convergent.

NUMERICAL EXAMPLE

Considering the variable coefficient space - time fractional advection-diffusion equation

$$\frac{\partial^a u(x,t)}{\partial t^\alpha} = \alpha(x)\frac{\partial^\beta u(x,t)}{\partial t^\beta} - b(x)\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} + q(x,t), 0 < x \le T,\ a(x) = \Gamma(2.5)x^{1.5}, b(x) = \Gamma(3.5)x^{0.5}, q(x,t) = x^2(1-x)\frac{t^{0.2}}{\Gamma(1.2)} + 2tx^2,$$

Where $0 < a, \gamma \le 1, 1 \le \beta < 2.$

We take $a = 0.8, \beta = 1.5, \gamma = 0.5, L = 1$, $u(x,0) = f(x) = 0, u(0,t) = u(1,t) = 0,$

$$u_{k+1}(x,t) = u_k(x,t) - \int_0^t (\frac{\partial^{0.8}}{\partial \xi^{0.8}} u_k(x,\xi) - \Gamma(2.5)x^{1.5}\frac{\partial^{1.5} u_k(x,\xi)}{\partial x^{1.5}} + \Gamma(3.5)x^{0.5}\frac{\partial^{1.5} u_k(x,\xi)}{\partial x^{1.5}}$$

Then this equation has exact solution $u(x,t) = x^2(1-x)t$. Next we use variation iteration method to compute this equations' approximate analytical solution.

By the initial conditions we get

$$u_0(x,t) = 0$$

$$u_1(x,t) = x^2(1-x)\frac{t^{1.2}}{\Gamma(2.2)} + x^2 t^2$$

$$u_2(x,t) = 2[x^2(1-x)\frac{t^{1.2}}{\Gamma(2.2)} + x^2 t^2] - x^2(1-x)\frac{t^{1.4}}{\Gamma(2.4)} - \frac{4x^2 t^{2.2}}{\Gamma(3.2)} - \frac{2}{3}x^2 t^3$$

Table 1: approximate analytical solution and exact solution

x	approximate analytical solution	exact solution
0.1000	0.0001	0.0001
0.2000	0.0003	0.0003
0.3000	0.0006	0.0006
0.4000	0.0010	0.0010
0.5000	0.0013	0.0013
0.6000	0.0015	0.0014
0.7000	0.0015	0.0014
0.8000	0.0014	0.0013
0.9000	0.0010	0.0008
1.0000	0.0002	0.0000

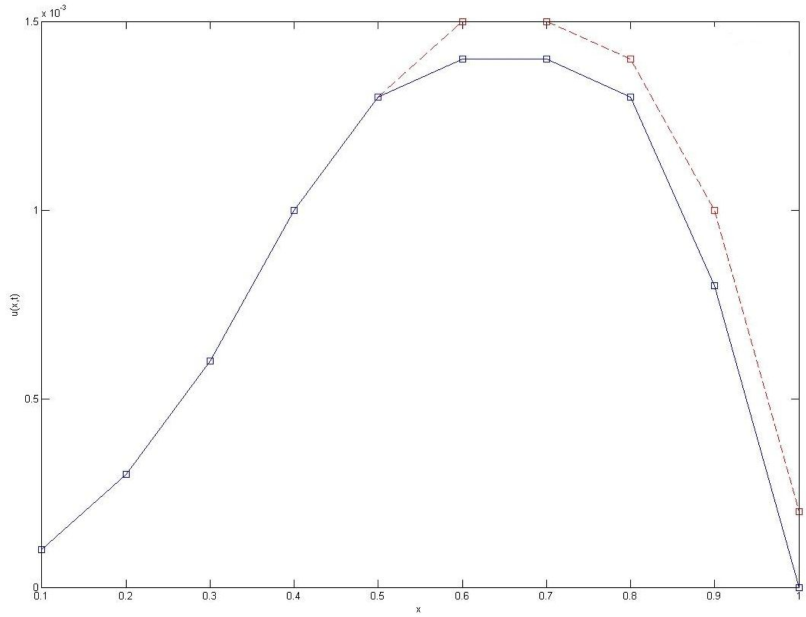

Figure 1: t=0.01 plan of approximate analytical solution and exact solution

From the table and figure, we get that the approximate analytical solution is consistent with the exact solution. So this method is effective.

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