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Research Article

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Research of 2D variable coefficient elliptic boundary value problem

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ABSTRACT

This paper used the finite element method and the finite difference method to solve the two-dimensional variable coefficient elliptic boundary value problem. It get the corresponding error analysis, and numerical simulation, the process requires application software MATLAB programming. Results show that they compared to the two methods and finite element the procedure is simple. The numerical results have the high accuracy and good stability. A two-dimensional variable coefficient elliptic partial differential equation is studied. This paper studies application of genetic algorithm to solve the genetic algorithm. A population search strategy basically does not depend on the knowledge of space or other auxiliary information, and the change rules of probability to control the search direction. It get the corresponding error analysis, and numerical simulation is carried out, the process of programming also requires the application of MATLAB software; results show that the nonlinear optimization method genetic algorithm does not need to choose the initial values of the results with high precision.

Keywords: elliptic equation; partial differential equation; difference method; finite element method

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INTRODUCTION

1. Background

In many areas (such as: Physics, mechanics, acoustics, electromagnetics, heat transferring etc.) and engineering technology, many problems can be described by differential equations. The differential equation is a powerful tool to describe and characterize the physical process, the system state, social and biological phenomenon, is one of the main ways of Mathematical Science and practice. The central task of differential equations is to meet the primary differential equation solution, boundary value conditions the solution. Especially in recent years, in resource exploration, aerospace engineering, earth physics, atmospheric measurements, marine engineering, biological organ morphology analysis, genetic engineering, quantum mechanics, elastic mechanics and the natural science and engineering technology have put forward a lot of differential equation problem. These examples illustrate the application of the numerical solution of partial differential equations in various subjects and in engineering. The solution of differential equations has become the core content of scientific and engineering computing. With the development of the computer itself, calculation method for the numerical solution of equations has also had the very big development, the development of the computational ability of the people is very important. Differential equations of the field is very broad, it from various backgrounds, application of the theory of category belongs to multi science, both in theoretical study and practical application significance are of great significance[1].

Solution of partial differential equations in general there is infinitely many, but to solve specific physical problems, must choose the solution which needs, from it, also must know the additional condition. Because partial differential equation is a representation of a kind of phenomenon of the common law, just know that the common law is not enough to master and understand the particularity of specific problems, so the physical phenomenon, particularity of each specific issue is specific conditions of the, is the initial and boundary conditions. Astronomy is similar, if the predicted motion of bodies, must know the quality of these objects, and in addition to the general formula of Newton's law, also need to know the initial state of celestial systems we study, is in a starting time, the distribution of these objects and their speed. When solve any mathematical physics equation, there are similar additional conditions. In mathematics, the initial conditions and boundary conditions are called boundary conditions. The partial differential equation itself is to express the same kind of physical phenomena are common, as a solution to the problem on the basis of; boundary conditions but reflect specific problems of personality, it puts forward the specific circumstances of the problem [2].

2 Finite element method

Finite element method first successful used in structural mechanics and solid mechanics, and later applied to the subject of elastic mechanics and fluid mechanics, etc.. The finite element method (finite element method) is an efficient computing method, commonly used. The finite element method in the early stage is based on the variation principle developed, so it is widely used in all kinds of physical field from Laplace equation and Poisson equation (described in this field and functional extreme problems are closely linked). Since 1969, some scholars apply the Galerkin method in the weighted residual method in fluid mechanics (Galerkin) or a least squares finite element equation is also obtained, so the finite element method can be applied to various types of physical field in any differential equations, and the extreme problems not to solve this kind of physical field and the functional have contact.

The finite element method, the earliest can be traced back to the 40 years of the twentieth Century. Courant was the first to use defined in the triangular area of the piecewise continuous function and the principle of minimum potential energy to solve the St.Venant torsion problem. The first successful attempt to modern finite element method was in 1956, Turner, Clough and others in the analysis of aircraft structure, the steel frame displacement method is applied to plane problem of mechanics of elasticity, the triangle elements for plane stress obtained correct answer problem are given[3]. In 1960, Clough further processing the plane elasticity problem, and for the first time put forward the "finite element method", make people aware of its effect.

The late 50's early 60's, computational mathematics China just starting soon, in isolated from the outside circumstances, Feng Kang led a group of scientific and technological personnel out from practice to theory, then from theory to practice Chinese computational mathematics development road to success. The research done on the engineering design of a lot of large elliptic equation analysis calculation problem, has accumulated rich experience and efficient. Feng Kang to summarize, made theoretical results system. In 1965, Feng Kang in the "Applied Mathematics and Computational Mathematics" published papers on "variational principle based differential format", is China independent from Western systematically introduced finite element method logo.

2.1 The principle of finite element method

The solution domain is discretized continuous as the combination of a group of units, with approximate function hypothesis in each cell to piecewise representation of the unknown field function domain to solve the approximation function, usually expressed by the unknown field function and its derivative numerical interpolation function at each node unit. It makes a continuous infinite degree of freedom problem into discrete finite degree of freedom the problem.

The finite element method of triangulation is introduced. The solution domain is discretized into finite elements. According to the relationship between basic variables and coordinates with one-dimensional, two-dimensional, three-dimensional decision unit. One dimensional line segment representation; two dimensional element can be triangular element goods quadrilateral element; 3D element commonly used tetrahedral or hexahedral element. Unit division more dense, the calculation precision is high, but the amount of computation is also more. Usually, the presence of variable changes can be a cell density, whereas the sparse [4].

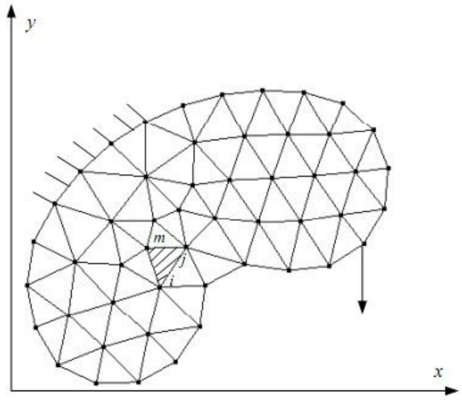

Fig.1 The node displacement

For any unit for any unit (i, j, m) that the node displacement to node displacement for the undetermined coefficient. It can interpolation functions is the unit. The solution of linear algebraic equations is very easy in mathematics.

That is to say the finite element method through the principle of unit discrete and the principle of minimum potential energy minimum potential energy, avoiding the differential equation directly for avoiding differential equation solving the mathematical difficulties, the solution of differential equations under the conditions of the converted into linear equations of the operation, implement any complex elasticity problems easily analysis and calculation [5].

The segmentation of unknown function at any point unit in the expansion function shape the segmentation unit function and discrete grid points on the value, namely the establishment of a linear interpolation function. In the finite element method, any point unit (x, y, z) of the field variables required by the selected interpolation obtained from the unit node value, i.e.

$$\Phi^{(e)}(x,y)=\Psi\Phi^{(e)}=\Psi_1(x,y)\Phi_1^{(e)}+\Psi_2(x,y)\Phi_2^{(e)}+\Psi_3(x,y)\Phi_3^{(e)}$$

Ψ is known as the element shape function matrix, it is related with the number of nodes and the interpolation of the element nodal coordinates. The number of shape function matrix component and element nodal degrees of freedom. The three node triangular element two-dimensional problem as an example, let each node has only one degree of freedom, any point unit (x, y) of the field $\Phi^{(e)}(x,y)$ can be expressed as follows:

$$\Phi^{(e)}(x,y)=\Psi\Phi^{(e)}=\Psi_1(x,y)\Phi_1^{(e)}+\Psi_2(x,y)\Phi_2^{(e)}+\Psi_3(x,y)\Phi_3^{(e)}$$

The basic steps of the finite element method to solve the problem as follows：
1) Integral equations are established, according to the variational principle or equation allowance and weight function orthogonality principle, establishment and differential equation with initial and boundary value problem of integral expressions of equivalent, which is the starting point of the finite element method [6].

2) Regional element subdivision, according to the physical characteristics of the shape of the area and to solve practical problems, the region is divided into several connected, non overlapping unit. Division of the regional unit is implemented using the finite element method of the preparatory work, the workload is relatively large, in addition to the calculation of element and node number and determine the relationship between, also said the positions of nodes, but also need to node number list the natural boundary and essential boundary conditions and the corresponding boundary value.

3) The determining unit basis function, according to the number of nodes in the approximate solution of unit and accuracy requirements, select the interpolation functions satisfying certain interpolation conditions as a unit basis function. Basic function of a finite element method is chosen in the unit, the unit is in regular geometric shape, in the selection of basis function can follow certain rule.

4) Unit analysis: the solving function of each unit with a linear combination of basic functions to approximate expression unit; then the approximate function into the integral equation, and integral to the unit area, can obtain with undetermined coefficient (i.e. the parameters of each node unit value) of algebraic equations, called the finite element equation.

5) The overall synthesis: after the finite element equation, the region all finite element equation according to a certain rule to accumulate, forming finite element equations in general.

6) The treatment of boundary conditions: general boundary conditions are of three types, divided into the essential boundary conditions (Dirichlet boundary conditions), the natural boundary conditions (Riemann boundary conditions), mixed boundary conditions (Cauchy boundary conditions). The natural boundary conditions, generally in the integral expressions can be automatically satisfied. The essential boundary conditions and mixed boundary conditions, according to a certain method of finite element equations are modified to meet the overall.

7) For solving finite element equations: according to the general finite element equations with boundary conditions is modified, the closed equations including all pending an unknown quantity, using appropriate numerical method, function can be the value of the node.

2.2 Mathematical model of finite element method
Consider the following boundary value problems：

$$\begin{cases} -\dfrac{\partial}{\partial x}\left[a(x,y)\dfrac{\partial u}{\partial x}\right]-\dfrac{\partial}{\partial y}\left[a(x,y)\dfrac{\partial u}{\partial y}\right]+\dfrac{\partial}{\partial x}(b_1(x,y)u) & \\ +\dfrac{\partial}{\partial y}(b_2(x,y)u)+c(x,y)u=f(x,y) & (x,y)\in\Omega \\ u(x,y)=0 & (x,y)\in\Gamma_1 \\ a\dfrac{\partial u(x,y)}{\partial n}+\sigma u(x,y)=\beta & (x,y)\in\Gamma_2 \end{cases} \tag{1}$$

The coefficient has much different physical significance in practical application: $a(x,y)$ is infiltration parameters; $b_1(x,y), b_2(x,y)$ is potential function; $f(x,y)$ source function; $\Gamma_1\cap\Gamma_2=\theta, \Gamma_1\cup\Gamma_2=\partial\Omega$ is regional boundary of Ω, θ is the empty set.

2.3 The discrete model of finite element method
Weak form derived solutions corresponding. Subspace and definition to introduce water Lev space $H^{'}(\Omega)$ on a bilinear function and linear function of $V=\left\{v\in H^{'}(\Omega)\middle| v|_{\Gamma_1}=0\right\}$, respectively:

$$F(v)=\iint_{\Omega} fvdxdy+\int_{\Gamma_2}\beta vds \tag{2}$$

That Galerkin meaning $u\in V$ is meet the generalized solution.

$$A(u,v)-F(v)=0, \forall v\in V \tag{2}$$

From the Galerkin method for approximate solutions is differential equations. Therefore, it constructs V a finite dimensional subspace V_h, then $u_k\in V_h$ for the following:

$$A(u_h,v)-F(v)=0, \forall v\in V_h \tag{3}$$

The general finite element on the regional is using a triangulation, and the rectangular partition with the two-dimensional area into a finite rectangular unit and make the number of nodes. May wish to set up each side of the rectangle are parallel and coordinate axis, and comply with the following rules：

a) No overlap between the internal rectangular；
b) Each vertex of the square, or the point on the boundary, or the common vertex adjacent rectangular.

Based on the function of the 3 tectonic units as follows:

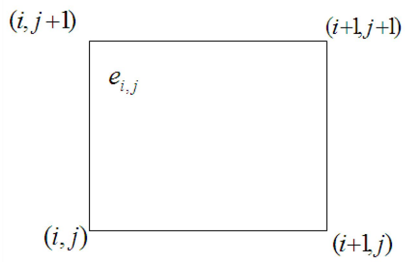

Fig.2 The 3 tectonic units

The unit V_h is expressed as follows:

$$V_h = span\{\Phi_{i,j}, \Phi_{i+1,j}, \Phi_{i+1,j+1}, \Phi_{i,j+1}\}$$

And u can be expressed as:

$$u = u_{i,j}\Phi_{i,j} + u_{i+1,j}\Phi_{i+1,j} + u_{i+1,j+1}\Phi_{i+1,j+1} + u_{i,j+1}\Phi_{i,j+1}$$

The formation of 4 element stiffness is matrix. Assembly of the total stiffness matrix is total load vector. Constraint handling can solve equations.

The error of finite element solution and the true solution:

$$|u - u_h|_0 \le ch|u - u_h|_1 \le c^2h^2|u|_2 \le c^2h^2|f|_0$$

3. Numerical simulation

3.1 Numerical examples one

Considering the boundary value problem which $\Omega = (0,1)\times(0,1)$, $a(x,y) = 1 + x + 2y^2$, $b_1 = 1 + 2y$, $b_2 = x + 1$, $c = x - y$. The true solution $u = xy(1-x)(1-y)$, $f(x,y)$ can be calculated.

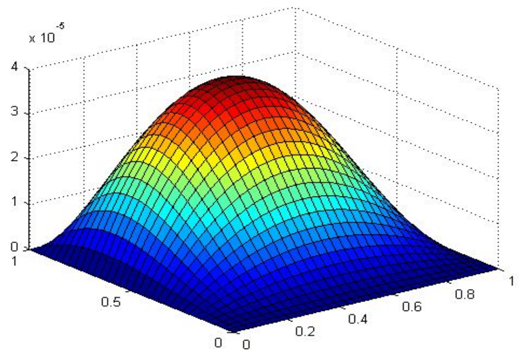

Fig.3 Error anatomic diagram when n=64

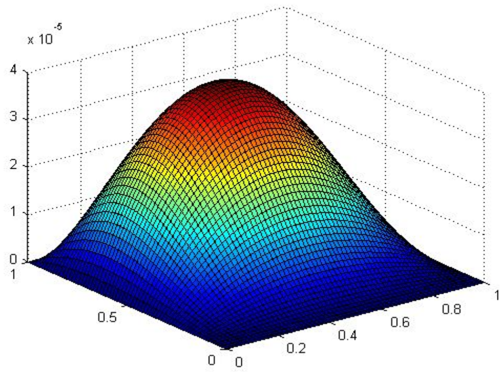

Fig.4 Error anatomic diagram when n=256

This example is solved. The true solution and numerical solution of the anatomical classification error below, the maximum error, but the relative error and the running time are shown in the table below:

n	The maximum error	Relative error	Run time
16	1.0330×10^{-4}	1.4545×10^{-3}	$0.1250(S)$
64	6.4866×10^{-6}	9.1111×10^{-5}	$0.7340(S)$
256	4.0548×10^{-7}	5.6952×10^{-6}	$13.5630(S)$

***3.2** Numerical examples two*

Considering the boundary value problem which $\Omega=(0,1)\times(0,1)$, $a(x,y)=1+x^2y$, $b_1=2+y$, $b_2=x+1$, $c=x+2y$. The true solution $u=\sin(\pi x)\sin(\pi y)$, $f(x,y)$ can be calculated by.

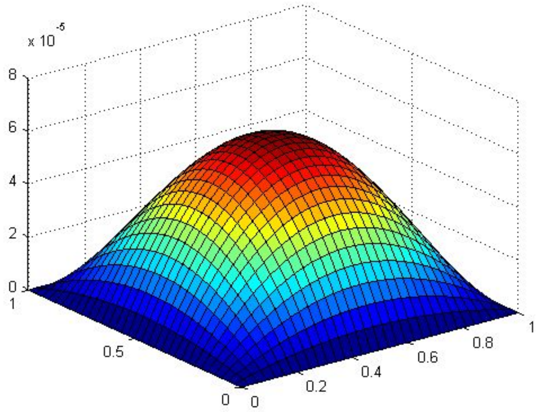

Fig.5 Error anatomic diagram when n=64

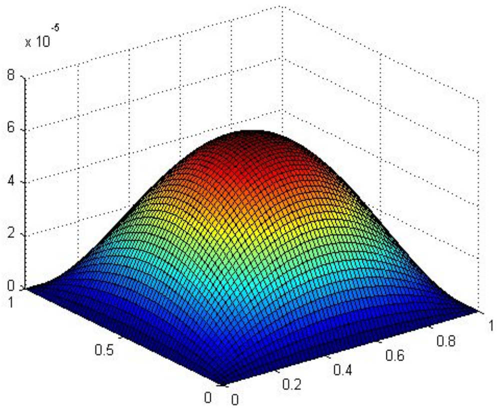

Fig.6 Error anatomic diagram when n=256

This example is solved. The true solution and numerical solution of the anatomical classification error below, the maximum error, but the relative error and the running time are shown in the table below:

n	The maximum error	Relative error	Run time
16	2.9648×10^{-3}	3.0637×10^{-3}	$0.2810(S)$
64	1.8573×10^{-4}	1.9165×10^{-4}	$0.7810(S)$
256	1.1610×10^{-5}	1.1979×10^{-5}	$12.2350(S)$

Summery

Elliptic equations exist widely in natural science and engineering technology, is a comprehensive research project with the interdisciplinary nature, high precision theory and algorithm research is important content in present many mathematical workers study. This study has enriched and developed the content. In this paper, the application of finite element method for solving the two-dimensional elliptic partial differential equation boundary value problem, the numerical results of high precision, good stability.

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