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On global smooth solutions for quantum Zakharov system

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ABSTRACT

This paper considers the existence and uniqueness of the global smooth solution for the initial value problem of quantum Zakharov system. By means of a priori integral estimates, Galerkin method and compactness theory, one has the existence and uniqueness.ct.

Key words: Global solutions; Guantum Zakharov system; Initial value problem

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INTRODUCTION

The Zakharov equations, derived by Zakharov in 1972 [1], describes the propagation of Langmuir waves in an unmagnetized plasma. The usual Zakharov system defined in space time $\mathbb{R}^{d+1}$ is given by

$$iE_t + \Delta E = nE,$$
$$n_{tt} - \Delta n = \Delta |E|^2,$$

Where $E:\mathbb{R}^{d+1} \to \mathbb{C}^d$ is the slowly varying amplitude of the high-frequency electric field, and $n:\mathbb{R}^{d+1} \to \mathbb{R}$ denotes the fluctuation of the ion-density from its equilibrium.

The Zakharov system was studied by many authors [2-6]. In [4], B. Guo , J. Zhang and X. Pu established globally in time existence and uniqueness of smooth solution for a generalized Zakharov equation in two dimensional case for small initial data, and proved global existence of smooth solution in one spatial dimension without any small assumption for initial data. Linares F. and Matheus C. [5] obtained global well-posedness results for the initial-value problem associated to the 1D Zakharov-Rubenchik system, and the results are sharp in some situations by proving ill-posedness results otherwise. In [6], Linares F. and Saut JC. proved that the Cauchy problem for the three-dimensional Zakharov-Kuznetsov equation is locally well-posed for data in $H^s(\mathbb{R}^3), s > \frac{9}{8}$.

By using of a quantum fluid approach, the following modified Zakharov equations are obtained [7].

$$iE_t + E_{xx} - H^2 E_{xxxx} = nE, \tag{0.1}$$
$$n_{tt} - n_{xx} + H^2 n_{xxxx} = |E|^2_{xx}, \tag{0.2}$$

Where H is the dimensionless quantum parameter given by the ratio of the ion plasmon and electron thermal energies. The presence of large value of H points to the possible experimental manifestation of quantum effects in the coupling between Langmuir and ion-acoustic modes in dense plasmas. Quantum effects may imply important consequences in the behavior of high density astrophysical plasmas. In this case, quantum effects cause an overall reduction in the wave-wave interaction level. This focusing effect may extend to quite long periods of time, indicating that the recurrence properties verified in the classical Zakharov equation are enhanced by the quantum effects.

In this paper, we study the systems of (1.1),(1.2) with the following initial value data.

$$E|_{t=0} = E_0(x),\ \ n|_{t=0} = n_0(x),\ n_t|_{t=0} = n_1(x),\ x \in \mathbb{R}, \tag{0.3}$$

We mainly consider the existence and uniqueness of the solution to the system.

We introduce a real-valued unknown function $\varphi(x,t)$. Then the problem (1.1)-(1.3) can be written in the form

$$iE_t + E_{xx} - H^2 E_{xxxx} - nE = 0, \tag{0.4}$$

$$n_t - \varphi_{xx} = 0, \tag{0.5}$$

$$\varphi_t - n + H^2 n_{xx} - |E|^2 = 0, \tag{0.6}$$

with initial data

$$E|_{t=0} = E_0(x),\ n|_{t=0} = n_0(x),\ \varphi|_{t=0} = \varphi_0(x). \tag{0.7}$$

Now we state the main results of the paper.

Theorem 1 Suppose that $E_0(x) \in H^{l+4}(\mathbb{R})$, $n_0(x) \in H^{l+4}(\mathbb{R})$, $n_1(x) \in H^{l+2}(\mathbb{R})$, $l \geq 1$. Then there exists a unique global smooth solution of the initial value problem (1.1)-(1.3).

$$E(x,t) \in L^\infty(0,T;H^{l+4}), \quad E_t(x,t) \in L^\infty(0,T;H^l)$$

$$n(x,t) \in L^\infty(0,T;H^{l+4}), \quad n_t(x,t) \in L^\infty(0,T;H^{l+2})$$

$$n_{tt}(x,t) \in L^\infty(0,T;H^l).$$

For the sake of convenience of the following contexts, we set some notations. For $1 \leq q \leq \infty$, we denote $L^q(\mathbb{R}^d)$ the space of all q times integrable functions in $\mathbb{R}^d$ equipped with norm $\|\cdot\|_{L^q(\mathbb{R}^d)}$ or simply $\|\cdot\|_{L^q}$ and $H^{s,p}(\mathbb{R}^d)$ the Sobolev space with norm $\|\cdot\|_{H^{s,p}(\mathbb{R}^d)}$. If $p = 2$, we write $H^s(\mathbb{R}^d)$ instead of $H^{s,2}(\mathbb{R}^d)$. Let $(f,g) = \int_{\mathbb{R}^n} f(x) \cdot \overline{g(x)} dx$, where $\overline{g(x)}$ denotes the complex conjugate function of $g(x)$.

The paper is organized as follows: In section 2, we make a priori estimates of the problem (1.4)-(1.7). In section 3, first of all, we obtain the existence and uniqueness of the global generalized solution of the problem (1.4)-(1.7) by Galerkin method. Next, the existence and uniqueness of the global smooth solution of the problem are obtained.

PRIORI ESTIMATIONS

Gl In this section, we will derive a priori estimations for the solution of the system (1.4)-(1.7).

Lemma 1 Suppose that $E_0(x) \in L^2(\mathbb{R})$. Then for the solution of Problem (1.4)-(1.7) we have

$$\|E\|_{L^2}^2 = \|E_0(x)\|_{L^2}^2.$$

Proof Taking the inner product of (1.4) and E, it follows that

$$(iE_t + E_{xx} - H^2 E_{xxxx} - nE, E) = 0 \tag{0.8}$$

since

$$\mathrm{Im}(iE_t, E) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E\|_{L^2}^2, \quad \mathrm{Im}(E_{xx} - H^2 E_{xxxx} - nE, E) = 0$$

hence from (2.1) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|E\|_{L^2}^2 = 0,$$

i.e.

$$\| E\|_{L^2}^2 = \| E_0(x)\|_{L^2}^2.$$

Remark Throughout Section 2 the inner products are taken here formally, since we have not yet established existence in the proper spaces.

Lemma 2 (Gagliardo-Nirenberg inequlity [8]) Assume that $u \in L^q(\mathbb{R}^n)$, $D^m u \in L^r(\mathbb{R}^n)$, $1 \le q, r \le \infty$, $0 \le j \le m$, we have the estimations

$$\| D^j u\|_{L^p(\mathbb{R}^n)} \le C \| D^m u\|_{L^r(\mathbb{R}^n)}^{\alpha} \| u\|_{L^q(\mathbb{R}^n)}^{1-\alpha},$$

where C is a positive constant, $0 \le \frac{j}{m} \le \alpha \le 1$,

$$\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\alpha)\frac{1}{q}.$$

Lemma 3 (Gronwall's inequality) Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies for a.e. t the differential inequality

$$\eta'(t) \le \phi(t)\eta(t) + \psi(t),$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0,T]$. Then

$$\eta(t) \le e^{\int_0^t \psi(s\tau)\mathrm{d}\tau}\left[\eta(0) + \int_0^t \psi(\tau)\mathrm{d}\tau\right]$$

for all $0 \le t \le T$.

Lemma 4 Suppose that $E_0(x) \in H^4(\mathbb{R})$, $n_0(x) \in H^2(\mathbb{R})$, $\varphi_0(x) \in H^2(\mathbb{R})$. Then we have

$$\sup_{0\le t\le T}\left[\| E_{xxxx}\|_{L^2}^2 + \| n_{xx}\|_{L^2}^2 + \| \varphi_{xx}\|_{L^2}^2 + \| E_t\|_{L^2}^2 + \| n_t\|_{L^2}^2 + \| \varphi_t\|_{L^2}^2\right] \le C_1$$

Proof Taking the inner products of (1.4) and $-E_t$, it follows that

$$\left(iE_t + E_{xx} - H^2 E_{xxxx} - nE, -E_t\right) = 0. \tag{0.9}$$

Since

$$\mathrm{Re}(iE_t, -E_t) = 0, \quad \mathrm{Re}(E_{xx}, -E_t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| E_x\|_{L^2}^2,$$

$$\mathrm{Re}(-H^2 E_{xxxx}, -E_t) = \frac{H^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| E_{xx}\|_{L^2}^2,$$

$$
\begin{aligned}
\mathrm{Re}(-nE,-E_t) &= \frac{1}{2}\int_{\text{i}} n\,|E|_t^2\,\mathrm{d}x \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\text{i}} n\,|E|^2\,\mathrm{d}x - \frac{1}{2}\int_{\text{i}} n_t\,|E|^2\,\mathrm{d}x \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\text{i}} n\,|E|^2\,\mathrm{d}x - \frac{1}{2}\int_{\text{i}} n_t(\varphi_t - n + H^2 n_{xx})\mathrm{d}x \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\text{i}} n\,|E|^2\,\mathrm{d}x - \frac{1}{2}\int_{\text{i}} n_t\varphi_t\mathrm{d}x + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|n\|_{L^2}^2 + \frac{H^2}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|n_x\|_{L^2}^2 \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\text{i}} n\,|E|^2\,\mathrm{d}x - \frac{1}{2}\int_{\text{i}} \varphi_{xx}\varphi_t\mathrm{d}x + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|n\|_{L^2}^2 + \frac{H^2}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|n_x\|_{L^2}^2 \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\text{i}} n\,|E|^2\,\mathrm{d}x + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_x\|_{L^2}^2 + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|n\|_{L^2}^2 + \frac{H^2}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|n_x\|_{L^2}^2,
\end{aligned}
$$

thus from (2.2) it follows that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|E_x\|_{L^2}^2 + H^2\|E_{xx}\|_{L^2}^2 + \int_{\text{i}} n\,|E|^2\,\mathrm{d}x + \frac{1}{2}\|\varphi_x\|_{L^2}^2 + \frac{1}{2}\|n\|_{L^2}^2 + \frac{H^2}{2}\|n_x\|_{L^2}^2\Big] = 0. \tag{0.10}
$$

Letting

$$
w(t) = \|E_x\|_{L^2}^2 + H^2\|E_{xx}\|_{L^2}^2 + \int_{\text{i}} n\,|E|^2\,\mathrm{d}x + \frac{1}{2}\|\varphi_x\|_{L^2}^2 + \frac{1}{2}\|n\|_{L^2}^2 + \frac{H^2}{2}\|n_x\|_{L^2}^2,
$$

and noticing (2.3) it follows that

$$
\mathrm{w}(\mathrm{t}) = \mathrm{w}(0). \tag{0.11}
$$

By using Hölder and Sobolev inequality, it follows that

$$
\begin{aligned}
\left|\int_{\text{i}} n\,|E|^2\,\mathrm{d}x\right| &\le \frac{1}{4}\|n\|_{L^2}^2 + \|E\|_{L^4}^4 \\
&\le \frac{1}{4}\|n\|_{L^2}^2 + C\|E_x\|_{L^2}\|E\|_{L^2}^3 \\
&\le \frac{1}{4}\|n\|_{L^2}^2 + \frac{1}{2}\|E_x\|_{L^2}^2 + C.
\end{aligned}
$$

Hence from (2.4) we get

$$
\|E_x\|_{L^2}^2 + \|E_{xx}\|_{L^2}^2 + \|\varphi_x\|_{L^2}^2 + \|n\|_{L^2}^2 + \|n_x\|_{L^2}^2 \le C.
$$

Taking the inner products of (1.6) and φ_{xxxx}, it follows that

$$
(\varphi_t - n + H^2 n_{xx} - |E|^2, \varphi_{xxxx}) = 0. \tag{0.12}
$$

Since

$$
\begin{aligned}
(\varphi_t, \varphi_{xxxx}) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_{xx}\|_{L^2}^2, \\
(-n, \varphi_{xxxx}) = (-n, n_{xxt}) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n_x\|_{L^2}^2, \\
(H^2 n_{xx}, \varphi_{xxxx}) = (H^2 n_{xx}, n_{xxt}) &= \frac{H^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n_{xx}\|_{L^2}^2,
\end{aligned}
$$

$$\left|(-|E|^2,\varphi_{xxxx})\right|=\left|(|E|^2_{xx},\varphi_{xx})\right|$$
$$\le C(\|E\|_{L^\infty}\|E_{xx}\|_{L^2}+\|E_x\|_{L^\infty}\|E_x\|_{L^2})\|\varphi_{xx}\|_{L^2}$$
$$\le C(\|\varphi_{xx}\|^2_{L^2}+1),$$

thus from (2.5) it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\varphi_{xx}\|^2_{L^2}+\|n_x\|^2_{L^2}+H^2\|n_{xx}\|^2_{L^2}\right]\le C(\|\varphi_{xx}\|^2_{L^2}+1).$$

(0.13)

By using Gronwall's inequality, it follows that

$$\|\varphi_{xx}\|^2_{L^2}+\|n_x\|^2_{L^2}+\|n_{xx}\|^2_{L^2}\le C.$$

From (1.5) and (1.6) we get

$$\|n_t\|_{L^2}\le C\|\varphi_{xx}\|_{L^2}\le C,$$
$$\|\varphi_t\|_{L^2}\le C(\|n\|_{L^2}+\|n_{xx}\|_{L^2}+\|E\|_{L^\infty}\|E\|_{L^2})\le C.$$

Differentiating (1.4) with respect to t, then taking the inner products of the resulting equation and E_t, it follows that

$$(iE_{tt}+E_{xxt}-H^2E_{xxxxt}-n_tE-nE_t,E_t)=0. \tag{0.14}$$

Since

$$\mathrm{Im}(iE_{tt},E_t)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E_t\|^2_{L^2},\quad \mathrm{Im}(E_{xxt}-H^2E_{xxxxt}-nE_t,E_t)=0,$$
$$\left|\mathrm{Im}(-n_tE,E_t)\right|\le C\|E\|_{L^\infty}\|n_t\|_{L^2}\|E_t\|_{L^2}\le C(\|E_t\|^2_{L^2}+1),$$

thus from (2.7) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|E_t\|^2_{L^2}\le C(\|E_t\|^2_{L^2}+1). \tag{0.15}$$

By using Gronwall's inequality, we get

$$\|E_t\|^2_{L^2}\le C.$$

From (1.4) we obtain

$$\|E_{xxxx}\|_{L^2}\le C(\|E_t\|_{L^2}+\|E_{xx}\|_{L^2}+\|n\|_{L^\infty}\|E\|_{L^2})\le C.$$

Lemma 5 Suppose that $E_0(x)\in H^4(¡)$, $n_0(x)\in H^4(¡)$, $\varphi_0(x)\in H^4(¡)$. Then we have

$$\sup_{0\le t\le T}\left[\|\varphi_{xxxx}\|^2_{L^2}+\|n_{xxxx}\|^2_{L^2}+\|\varphi_{xxt}\|^2_{L^2}+\|n_{xxt}\|^2_{L^2}+\|n_{tt}\|^2_{L^2}\right]\le C.$$

Proof taking the inner product of (1.6) and $\partial_x^8\varphi$, it follows that

$$(\varphi_t-n+H^2n_{xx}-|E|^2,\partial_x^8\varphi)=0. \tag{0.16}$$

Since

$$(\varphi_t,\partial_x^8\varphi)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_{xxxx}\|^2_{L^2},$$

$$(-n,\partial_x^8\varphi)=(-n,\partial_x^6 n_t)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| n_{xxx}\|_{L^2}^2,$$

$$(H^2 n_{xx},\partial_x^8\varphi)=(H^2 n_{xx},\partial_x^6 n_t)=\frac{H^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| n_{xxxx}\|_{L^2}^2,$$

$$\left|(-|E|^2,\partial_x^8\varphi)\right|=\left|(\partial_x^4|E|^2,\varphi_{xxxx})\right|\le C\| E\|_{L^2}\| E_{xxxx}\|_{L^2}\| \varphi_{xxxx}\|_{L^2}$$
$$\le C(\| \varphi_{xxxx}\|_{L^2}^2+1),$$

thus from (2.9) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\| \varphi_{xxxx}\|_{L^2}^2+\| n_{xxx}\|_{L^2}^2+\| n_{xxxx}\|_{L^2}^2\right]\le C(\| \varphi_{xxxx}\|_{L^2}^2+1). \tag{0.17}$$

By using Gronwall's inequality, it follows that

$$\| \varphi_{xxxx}\|_{L^2}^2+\| n_{xxx}\|_{L^2}^2+\| n_{xxxx}\|_{L^2}^2\le C.$$

From (1.5) and (1.6) we obtain

$$\| \varphi_{xxt}\|_{L^2}\le C(\| n_{xx}\|_{L^2}+\| n_{xxxx}\|_{L^2}+\| E\|_{L^2}\| E_{xx}\|_{L^2})\le C,$$
$$\| n_{xxt}\|_{L^2}\le C\| \varphi_{xxxx}\|_{L^2}\le C,$$
$$\| n_{tt}\|_{L^2}\le C\| \varphi_{xxt}\|_{L^2}\le C.$$

Lemma 6 Suppose that $E_0(x)\in H^{m+4}(\mathbb{R})$, $n_0(x)\in H^{m+2}(\mathbb{R})$, $\varphi_0(x)\in H^{m+2}(\mathbb{R}), m\ge 0$. Then we have

$$\sup_{0\le t\le T}\left[\| E(x,t)\|_{H^{m+4}}+\| n(x,t)\|_{H^{m+2}}+\| \varphi(x,t)\|_{H^{m+2}}+\| E_t(x,t)\|_{H^m}+\| n_t(x,t)\|_{H^m}+\| \varphi_t\|_{H^m}\right]\le C.$$

where the constant C only depends on $\| E_0(x)\|_{H^{m+4}}$, $\| n_0(x)\|_{H^{m+2}}$, $\| \varphi_0(x)\|_{H^{m+2}}$.

Proof The lemma 6 is true when $m=0$ (lemma 4). Suppose the lemma 6 is true when $m=k,(k\ge 0)$, i.e.

$$\sup_{0\le t\le T}\left[\| E(x,t)\|_{H^{k+4}}+\| n(x,t)\|_{H^{k+2}}+\| \varphi(x,t)\|_{H^{k+2}}+\| E_t(x,t)\|_{H^k}+\| n_t(x,t)\|_{H^k}+\| \varphi_t\|_{H^k}\right]\le C.$$

Taking the inner products of (1.6) and $(-1)^{k+1}\partial_x^{2(k+3)}\varphi$, it follows that

$$(\varphi_t-n+H^2 n_{xx}-|E|^2,(-1)^{k+1}\partial_x^{2(k+3)}\varphi)=0. \tag{0.18}$$

Since

$$(\varphi_t,(-1)^{k+1}\partial_x^{2(k+3)}\varphi)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| \partial_x^{k+3}\varphi\|_{L^2}^2,$$

$$(-n,(-1)^{k+1}\partial_x^{2(k+3)}\varphi)=(-n,(-1)^{k+1}\partial_x^{2(k+2)}n_t)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| \partial_x^{k+2}n\|_{L^2}^2,$$

$$(H^2 n_{xx},(-1)^{k+1}\partial_x^{2(k+3)}\varphi)=(H^2 n_{xx},(-1)^{k+1}\partial_x^{2(k+2)}n_t)=\frac{H^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| \partial_x^{k+3}n\|_{L^2}^2,$$

$$\left|(-|E|^2,(-1)^{k+1}\partial_x^{2(k+3)}\varphi)\right|=\left|(\partial_x^{k+3}|E|^2,\partial_x^{k+3}\varphi)\right|$$
$$\le C\| E\|_{L^2}\| \partial_x^{k+3}E\|_{L^2}\| \partial_x^{k+3}\varphi\|_{L^2}$$
$$\le C(\| \partial_x^{k+3}\varphi\|_{L^2}^2+1),$$

thus from (2.11) it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\partial_x^{k+3}\varphi\|_{L^2}^2+\|\partial_x^{k+2}n\|_{L^2}^2+H^2\|\partial_x^{k+3}n\|_{L^2}^2\right]\le C(\|\partial_x^{k+3}\varphi\|_{L^2}^2+1). \tag{0.19}$$

By using Gronwall's inequality, it follows that

$$\|\partial_x^{k+3}\varphi\|_{L^2}^2+\|\partial_x^{k+2}n\|_{L^2}^2+\|\partial_x^{k+3}n\|_{L^2}^2\le C.$$

From (1.5) and (1.6) we get

$$\|\partial_x^{k+1}n_t\|_{L^2}\le C\|\partial_x^{k+3}\varphi\|_{L^2}\le C,$$
$$\|\partial_x^{k+1}\varphi_t\|_{L^2}\le C(\|\partial_x^{k+1}n\|_{L^2}+\|\partial_x^{k+3}n\|_{L^2}+\|E\|_{L^\infty}\|\partial_x^{k+1}E\|_{L^2})\le C.$$

Differentiating (1.4) with respect to t, then taking the inner products of the resulting equation and $(-1)^{k+1}\partial_x^{2(k+1)}E_t$, it follows that

$$(iE_{tt}+E_{xxt}-H^2E_{xxxxt}-n_tE-nE_t,(-1)^{k+1}\partial_x^{2(k+1)}E_t)=0. \tag{0.20}$$

Since

$$\mathrm{Im}(iE_{tt},(-1)^{k+1}\partial_x^{2(k+1)}E_t)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^{k+1}E_t\|_{L^2}^2,$$
$$\mathrm{Im}(E_{xxt}-H^2E_{xxxxt},(-1)^{k+1}\partial_x^{2(k+1)}E_t)=0,$$
$$\begin{aligned}\left|\mathrm{Im}(-n_tE,(-1)^{k+1}\partial_x^{2(k+1)}E_t)\right|&\le\left|(\partial_x^{k+1}(n_tE),\partial_x^{k+1}E_t)\right|\\&\le C\left(\|\partial_x^{k+1}n_t\|_{L^2}\|E\|_{L^2}+\|\partial_x^{k+1}E\|_{L^2}\|n_t\|_{L^2}\right)\|\partial_x^{k+1}E_t\|_{L^2}\\&\le C(\|\partial_x^{k+1}E_t\|_{L^2}^2+1),\end{aligned}$$
$$\begin{aligned}\left|\mathrm{Im}(-nE_t,(-1)^{k+1}\partial_x^{2(k+1)}E_t)\right|&\le\left|(\partial_x^{k+1}(nE_t),\partial_x^{k+1}E_t)\right|\\&\le C\left(\|\partial_x^{k+1}n\|_{L^2}\|E_t\|_{L^2}+\|\partial_x^{k+1}E_t\|_{L^2}\|n\|_{L^2}\right)\|\partial_x^{k+1}E_t\|_{L^2}\\&\le C(\|\partial_x^{k+1}E_t\|_{L^2}^2+1),\end{aligned}$$

thus from (2.13) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^{k+1}E_t\|_{L^2}^2\le C(\|\partial_x^{k+1}E_t\|_{L^2}^2+1). \tag{0.21}$$

By using Gronwall's inequality, we get

$$\|\partial_x^{k+1}E_t\|_{L^2}^2\le C.$$

From (1.4) we obtain

$$\|\partial_x^{k+5}E\|_{L^2}\le C(\|\partial_x^{k+1}E_t\|_{L^2}+\|\partial_x^{k+3}E\|_{L^2}+\|\partial_x^{k+1}n\|_{L^2}\|E\|_{L^2}+\|n\|_{L^2}\|\partial_x^{k+1}E\|_{L^2})\le C.$$

Hence

$$\begin{aligned}\sup_{0\le t\le T}\Big[&\|E(x,t)\|_{H^{k+5}}+\|n(x,t)\|_{H^{k+3}}+\|\varphi(x,t)\|_{H^{k+3}}+\|E_t(x,t)\|_{H^{k+1}}+\\&+\|n_t(x,t)\|_{H^{k+1}}+\|\varphi_t\|_{H^{k+1}}\Big]\le C.\end{aligned}$$

The lemma 6 is proved completely.

Lemma 7 Suppose that $E_0(x)\in H^{l+4}(\mathbb{R})$, $n_0(x)\in H^{l+4}(\mathbb{R})$, $\varphi_0(x)\in H^{l+4}(\mathbb{R})$, $l\ge 0$. Then we have

$$\sup_{0\le t\le T}\Big[\| E(x,t)\|_{H^{l+4}} + \| n(x,t)\|_{H^{l+4}} + \| \varphi(x,t)\|_{H^{l+4}} + \| E_t(x,t)\|_{H^{l}} + \\ + \| n_t(x,t)\|_{H^{l+2}} + \| \varphi_t\|_{H^{l+2}} + \| n_{tt}(x,t)\|_{H^{l}}\Big] \le C.$$

where the constant C only depends on $\| E_0(x)\|_{H^{l+4}}$, $\| n_0(x)\|_{H^{l+4}}$, $\| \varphi_0(x)\|_{H^{l+4}}$.

Proof The lemma 7 is true when $l=0$ (lemma 4, 5). Suppose the lemma 7 is true when $l=k,(k\ge 0)$, i.e.

$$\sup_{0\le t\le T}\Big[\| E(x,t)\|_{H^{k+4}} + \| n(x,t)\|_{H^{k+4}} + \| \varphi(x,t)\|_{H^{k+4}} + \| E_t(x,t)\|_{H^{k}} + \\ + \| n_t(x,t)\|_{H^{k+2}} + \| \varphi_t\|_{H^{k+2}} + \| n_{tt}(x,t)\|_{H^{k}}\Big] \le C.$$

Differentiating (1.4) with respect to t, then taking the inner product of the resulting equation and $(-1)^{k+1}\partial_x^{2(k+1)}E_t$, it follows that

$$(iE_{tt} + E_{xxt} - H^2 E_{xxxxt} - n_t E - nE_t, (-1)^{k+1}\partial_x^{2(k+1)}E_t) = 0. \tag{0.22}$$

Since

$$\mathrm{Im}(iE_{tt}, (-1)^{k+1}\partial_x^{2(k+1)}E_t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| \partial_x^{k+1}E_t\|_{L^2}^2,$$
$$\mathrm{Im}(E_{xxt} - H^2E_{xxxxt}, (-1)^{k+1}\partial_x^{2(k+1)}E_t) = 0,$$
$$\left|\mathrm{Im}(-n_tE, (-1)^{k+1}\partial_x^{2(k+1)}E_t)\right| \le \left|(\partial_x^{k+1}(n_tE), \partial_x^{k+1}E_t)\right|$$
$$\le C(\| \partial_x^{k+1}n_t\|_{L^2}\| E\|_{L^2} + \| n_t\|_{L^2}\| \partial_x^{k+1}E\|_{L^2})\| \partial_x^{k+1}E_t\|_{L^2}$$
$$\le C(\| \partial_x^{k+1}E_t\|_{L^2}^2 + 1),$$
$$\left|\mathrm{Im}(-nE_t, (-1)^{k+1}\partial_x^{2(k+1)}E_t)\right| \le \left|(\partial_x^{k+1}(nE_t), \partial_x^{k+1}E_t)\right|$$
$$\le C(\| \partial_x^{k+1}n\|_{L^2}\| E_t\|_{L^2} + \| n\|_{L^2}\| \partial_x^{k+1}E_t\|_{L^2})\| \partial_x^{k+1}E_t\|_{L^2}$$
$$\le C(\| \partial_x^{k+1}E_t\|_{L^2}^2 + 1),$$

thus from (2.15) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| \partial_x^{k+1}E_t\|_{L^2}^2 \le C(\| \partial_x^{k+1}E_t\|_{L^2}^2 + 1). \tag{0.23}$$

By using Gronwall's inequality, it follows that

$$\| \partial_x^{k+1}E_t\|_{L^2}^2 \le C.$$

From (1.4) we obtain

$$\| \partial_x^{k+5}E\|_{L^2} \le C(\| \partial_x^{k+1}E_t\|_{L^2} + \| \partial_x^{k+3}E\|_{L^2} + \| \partial_x^{k+1}n\|_{L^2}\| E\|_{L^2} + \\ + \| n\|_{L^2}\| \partial_x^{k+1}E\|_{L^2}) \le C.$$
$$\| \partial_x^{k+5}E\|_{L^2} \le C(\| \partial_x^{k+1}E_t\|_{L^2} + \| \partial_x^{k+3}E\|_{L^2} + \| \partial_x^{k+1}n\|_{L^2}\| E\|_{L^2} + \\ + \| n\|_{L^2}\| \partial_x^{k+1}E\|_{L^2}) \le C.$$

Taking the inner product of (1.6) and $(-1)^{k+1}\partial_x^{2(k+5)}\varphi$, it follows that

$$(\varphi_t - n + H^2n_{xx} + |E|^2, (-1)^{k+1}\partial_x^{2(k+5)}\varphi) = 0. \tag{0.24}$$

Since

$$(\varphi_t,(-1)^{k+1}\partial_x^{2(k+5)}\varphi)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^{k+5}\varphi\|_{L^2}^2,$$

$$(-n,(-1)^{k+1}\partial_x^{2(k+5)}\varphi)=(-n,(-1)^{k+1}\partial_x^{2(k+4)}n_t)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^{k+4}n\|_{L^2}^2,$$

$$(H^2n_{xx},(-1)^{k+1}\partial_x^{2(k+5)}\varphi)=(H^2n_{xx},(-1)^{k+1}\partial_x^{2(k+4)}n_t)=\frac{H^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^{k+5}n\|_{L^2}^2,$$

$$\begin{aligned}\left|(|E|^2,(-1)^{k+1}\partial_x^{2(k+5)}\varphi)\right|&=\left|(\partial^{k+5}|E|^2,\partial_x^{k+5}\varphi)\right|\\&\le C\|\partial_x^{k+5}E\|_{L^2}\|E\|_{L^2}\|\partial_x^{k+5}\varphi\|_{L^2}\\&\le C(\|\partial_x^{k+5}\varphi\|_{L^2}^2+1),\end{aligned}$$

thus from (2.17) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\partial_x^{k+5}\varphi\|_{L^2}^2+\|\partial_x^{k+4}n\|_{L^2}^2+H^2\|\partial_x^{k+5}n\|_{L^2}^2\right]\le C(\|\partial_x^{k+5}\varphi\|_{L^2}^2+1). \tag{0.25}$$

By using Gronwall's inequality, it follows that

$$\|\partial_x^{k+5}\varphi\|_{L^2}^2+\|\partial_x^{k+4}n\|_{L^2}^2+\|\partial_x^{k+5}n\|_{L^2}^2\le C.$$

From (1.5) and (1.6) we get

$$\|\partial_x^{k+3}\varphi_t\|_{L^2}\le C(\|\partial_x^{k+3}n\|_{L^2}+\|\partial_x^{k+5}n\|_{L^2}+\|\partial_x^{k+3}E\|_{L^2}\|E\|_{L^2})\le C,$$

$$\|\partial_x^{k+3}n_t\|_{L^2}\le C\|\partial_x^{k+5}\varphi\|_{L^2}\le C,$$

$$\|\partial_x^{k+1}n_{tt}\|_{L^2}\le C\|\partial_x^{k+3}\varphi_t\|_{L^2}\le C.$$

Hence

$$\begin{aligned}\sup_{0\le t\le T}\Big[&\|E(x,t)\|_{H^{k+5}}+\|n(x,t)\|_{H^{k+5}}+\|\varphi(x,t)\|_{H^{k+5}}+\|E_t(x,t)\|_{H^{k+1}}+\\&+\|n_t(x,t)\|_{H^{k+3}}+\|\varphi_t\|_{H^{k+3}}+\|n_{tt}(x,t)\|_{H^{k+1}}\Big]\le C.\end{aligned}$$

The lemma 7 is proved completely.

THE EXISTENCE AND UNIQUENESS OF SOLUTION

In this section, we formulate the proof of theorem 1. First we give the definition of generalized solution for problem (1.4)-(1.7).

Definition 1 The functions $E(x,t)\in L^\infty(0,T;H^4(\mathbb{R}))\cap W^{1,\infty}(0,T;L^2(\mathbb{R}))$, $n(x,t)\in L^\infty(0,T;H^2(\mathbb{R}))$ $\cap W^{1,\infty}(0,T;L^2(\mathbb{R}))$, $\varphi(x,t)\in L^\infty(0,T;H^2(\mathbb{R}))\cap W^{1,\infty}(0,T;L^2(\mathbb{R}))$ are called generalized solution of problem (1.4)-(1.7), if they satisfy the following integral equality.

$$(iE_t,v)+(\partial_x^2E,v)-(H^2\partial_x^4E,v)-(nE,v)=0,$$

$$(n_t,v)-(\varphi_{xx},v)=0,$$

$$(\varphi_t,v)-(n,v)+(H^2n_{xx},v)-(|E|^2,v)=0,$$

$$(E(x,0),v)=(E_0(x),v),\quad (n(x,0),v)=(n_0(x),v),\quad (\varphi(x,0),v)=(\varphi_0(x),v).$$

Now, one can estimate the following theorem.

Theorem 2 Suppose that $E_0(x)\in H^4(\mathbb{R})$, $n_0(x)\in H^2(\mathbb{R})$, $\varphi_0(x)\in H^2(\mathbb{R})$. Then there exists a global generalized solution of the initial value problem (1.4)-(1.7),

$$E(x,t)\in L^\infty(0,T;H^4),\quad E_t(x,t)\in L^\infty(0,T;L^2)$$
$$n(x,t)\in L^\infty(0,T;H^2),\quad n_t(x,t)\in L^\infty(0,T;L^2)$$
$$\varphi(x,t)\in L^\infty(0,T;H^2),\quad \varphi_t(x,t)\in L^\infty(0,T;L^2).$$

Proof The existence is proved by the Faedo-Galerkin method. We choose the basic periodic functions $\{\omega_j(x)\}$ as follows

$$-\omega_j''(x)=\lambda_j\omega_j(x),\ \omega_j(x)\in H^2\cap H_0^1,\quad j=1,2,\ldots,l.$$

For each l we define an approximate solution (E_l,n_l,φ_l) of problem (1.4)-(1.7) as follows

$$E^l(x,t)=\sum_{j=1}^{l}\alpha_j(t)\omega_j(x),\quad n^l(x,t)=\sum_{j=1}^{l}\beta_j(t)\omega_j(x),\quad \varphi^l(x,t)=\sum_{j=1}^{l}\gamma_j(t)\omega_j(x),$$

and

$$i(E_t^l,\omega_s)+(\partial_x^2E^l,\omega_s)-(H^2\partial_x^4E^l,\omega_s)-(n^lE^l,\omega_s)=0, \tag{1.1}$$
$$(n_t^l,\omega_s)-(\partial_x^2\varphi^l,\omega_s)=0,\quad k=1,\ldots,d;\ s=1,\ldots,l, \tag{1.2}$$
$$(\varphi_t^l,\omega_s)-(n^l,\omega_s)+(H^2\partial_x^2n^l,\omega_s)-(|E^l|^2,\omega_s)=0, \tag{1.3}$$
$$E^l(x,0)=E_0^l(x),\quad n^l(x,0)=n_0^l(x),\quad \varphi^l(x,0)=\varphi_0^l(x), \tag{1.4}$$

and

$$E_0^l(x)\xrightarrow{H^4}E_0(x),\quad n_0^l(x)\xrightarrow{H^2}n_0(x),\quad \varphi_0^l(x)\xrightarrow{H^2}\varphi_0(x),\quad l\to\infty$$

Equations (3.1)-(3.4) are equivalent to an initial-value problem for a linear finite-dimensional ordinary differential equation for $\alpha_j(t)$, $\beta_j(t)$ and $\gamma_j(t)$. The existence and uniqueness is obvious.

Similarly to the proofs of lemma 1 and 4, for the approximate solution (E_l,n_l,φ_l), we can establish the following estimations

$$\sup_{0\le t\le T}\left[\|E^l(\cdot,t)\|_{H^4}+\|n^l(\cdot,t)\|_{H^2}+\|\varphi^l(\cdot,t)\|_{H^2}\right]\le C, \tag{1.5}$$
$$\sup_{0\le t\le T}\left[\|E_t^l(\cdot,t)\|_{L^2}+\|n_t^l(\cdot,t)\|_{L^2}+\|\varphi_t^l(\cdot,t)\|_{L^2}\right]\le C, \tag{1.6}$$

where the constants C is independent of l. By a compactness argument, we can choose subsequence $E^\nu(x,t)$, $n^\nu(x,t)$, $\varphi^\nu(x,t)$, such that $\nu\to\infty$

$$E^\nu(x,t)\to E(x,t)\ \text{ in } L^\infty(0,T;H^4)\ \text{ weakly star,}$$
$$E^\nu(x,t)\to E(x,t)\ \text{ in strong topology of } H^3(Q_T),$$
$$E_t^\nu(x,t)\to E_t(x,t)\ \text{ in } L^\infty(0,T;L^2)\ \text{ weakly star,}$$
$$n^\nu(x,t)\to n(x,t)\ \text{ in } L^\infty(0,T;H^2)\ \text{ weakly star,}$$
$$n^\nu(x,t)\to n(x,t)\ \text{ in strong topology of } H^1(Q_T),$$
$$n_t^\nu(x,t)\to n_t(x,t)\ \text{ in } L^\infty(0,T;L^2)\ \text{ weakly star,}$$
$$\varphi^\nu(x,t)\to\varphi(x,t)\ \text{ in } L^\infty(0,T;H^2)\ \text{ weakly star,}$$
$$\varphi^\nu(x,t)\to\varphi(x,t)\ \text{ in strong topology of } H^1(Q_T),$$
$$\varphi_t^\nu(x,t)\to\varphi_t(x,t)\ \text{ in } L^\infty(0,T;L^2)\ \text{ weakly star,}$$
$$n^\nu E^\nu\to nE\ \text{ in } L^\infty(0,T;L^2)\ \text{ weakly star,}$$
$$|E^\nu(x,t)|^2\to|E(x,t)|^2\ \text{ in } L^\infty(0,T;L^2)\ \text{ weakly star,}$$

where $Q_T=[0,T]\times\Omega$, $\bar{\Omega}=[-D,D]$. Hence taking $l=\nu\to\infty$ from (3.1)-(3.4), by using the density of ω_j

in $L^2(\Omega)$ we get the existence of local generalized solution for the periodic initial Problem (1.4)-(1.7). Similarly to the proof of [9], letting $D \to \infty$, the existence of local solution for the initial value Problem (1.4)-(1.7) can be obtain. By the continuation extension principle, from the conditions of the theorem and a priori estimates in Section 2, we can get the existence of global solution for Problem (1.4)-(1.7).
Next, we prove the uniqueness of solution.

Theorem 3 Suppose that the conditions of theorem 2 are satisfied. Then the global generalized solution of the initial value Problem (1.4)-(1.7) is unique.

Proof Suppose that there are two solutions E_1, n_1, φ_1 and E_2, n_2, φ_2. Let

$$E = E_1 - E_2, \quad n = n_1 - n_2, \quad \varphi = \varphi_1 - \varphi_2.$$

From (1.4)-(1.7) we get

$$iE_t + E_{xx} - H^2 E_{xxxx} - n_1 E_1 + n_2 E_2 = 0, \tag{1.7}$$

$$n_t - \varphi_{xx} = 0, \tag{1.8}$$

$$\varphi_t - n + H^2 n_{xx} - |E_1|^2 + |E_2|^2 = 0, \tag{1.9}$$

$$E|_{t=0} = 0, \quad n|_{t=0} = 0, \quad \varphi|_{t=0} = 0, \quad x \in ¡ \tag{1.10}$$

Taking the inner product of (3.7) and E, it follows that

$$(iE_t + E_{xx} - H^2 E_{xxxx} - n_1 E_1 + n_2 E_2, E) = 0. \tag{1.11}$$

Since

$$\mathrm{Im}(iE_t, E) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| E\|_{L^2}^2, \quad \mathrm{Im}(E_{xx} - H^2 E_{xxxx}, E) = 0,$$

$$\left|\mathrm{Im}(n_1 E_1 - n_2 E_2, E)\right| \le \left|(nE_1 - n_2 E, E)\right|$$
$$\le C(\| E_1\|_{L^\infty} \| n\|_{L^2} + \| n_2\|_{L^\infty} \| E\|_{L^2}) \| E\|_{L^2}$$
$$\le C(\| n\|_{L^2}^2 + \| E\|_{L^2}^2)$$

where the first successive equality result from Lions' Lemma. Thus from (3.11) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\| E\|_{L^2}^2 \le C(\| n\|_{L^2}^2 + \| E\|_{L^2}^2). \tag{1.12}$$

Taking the inner product of (3.9) and φ, it follows that

$$(\varphi_t - n + H^2 n_{xx} - |E_1|^2 + |E_2|^2, \varphi) = 0. \tag{1.13}$$

Since

$$(\varphi_t, \varphi) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| \varphi\|_{L^2}^2,$$

$$\left|(-n, \varphi)\right| \le C(\| n\|_{L^2}^2 + \| \varphi\|_{L^2}^2),$$

$$(H^2 n_{xx}, \varphi) = (H^2 n, \varphi_{xx}) = (H^2 n, n_t) = \frac{H^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| n\|_{L^2}^2,$$

$$\left|(-|E_1|^2 + |E_2|^2, \varphi)\right| = \left|((E_1 - E_2)\overline{E_1} + E_2(\overline{E_1} - \overline{E_2}), \varphi)\right|$$
$$\le C(\| E_1\|_{L^\infty} \| E\|_{L^2} \| + \| E_2\|_{L^\infty} \| E\|_{L^2}) \| \varphi\|_{L^2}$$
$$\le C(\| E\|_{L^2}^2 + \| \varphi\|_{L^2}^2),$$

where the first successive equality result from Lions' Lemma. Thus from (3.13) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\| \varphi\|_{L^2}^2 + H^2 \| n\|_{L^2}^2\right] \le C(\| E\|_{L^2}^2 + \| n\|_{L^2}^2 + \| \varphi\|_{L^2}^2). \tag{1.14}$$

Hence from (3.12) and (3.14) and we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\| E\|_{L^2}^2 + H^2\| n\|_{L^2}^2 + \| \varphi\|_{L^2}^2\right] \le C\left[\| E\|_{L^2}^2 + H^2\| n\|_{L^2}^2 + \| \varphi\|_{L^2}^2\right].$$

By using Gronwall's inequality and noticing (3.10), it follows that

$$E \equiv 0, \quad n \equiv 0, \quad \varphi \equiv 0.$$

The theorem is proved.

Using lemma 6, 7 and the embedding theorems of Sobolev spaces, the result of theorem 1 is obvious.

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