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Research Article

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Implicit bi-diagonal scheme for computation of discontinuous flow problems

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ABSTRACT

This paper is concerned with a mathematical model for numerical simulation of 1D discontinuous flow problems. The governing water equations are solved by an implicit bi-diagonal numerical scheme, based on the MacCormack's predictor-corrector technique, and the oscillations near the vicinity of discontinuity/shocks are smoothed by a proposed technique. The mathematical model was used to numerically compute the water surface from supercritical flow to subcritical one or from subcritical flow to supercritical one in a rectangular open channel, and was also used to predict 1D wave due to an instantaneous opening of sluice gate problem in a rectangular open channel, and their results are in agreement with the theoretical results, which shows that the proposed method is accurate, reliable and effective in simulation of discontinuous flow problems.

Keywords: numerical simulation, implicit scheme, oscillation, discontinuous flow problems, predictor-corrector technique

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INTRODUCTION

The discontinuous flow problems such as flood waves often appear in hydraulic engineering. The flood wave caused by a dam failure can result in the loss of human lives and have a severe economic impact. Therefore, significant efforts have been carried out over the years to produce methods for determination of the extent and timing of the flood wave. Most of these methods are based on the solution of the Shallow Water equations. By following this approach, the originally three-dimensional problem reduces to 2D Shallow Water equations (1D case described by the Saint Venant equations) with a moving boundary separating the wet and dry areas of the domain. One feature of hyperbolic equations of this type is the formation of bores (i.e., the rapidly varying discontinuous flow). It is an important basis for validating the numerical method whether the scheme can capture the bore waves accurately or not. This gives rise to an increasing interest in solving such a problem.

To overcome these problems and improve shock capturing property of finite difference schemes, various good shock-capturing finite-difference methods developed and applied widely in computational gas dynamics have been introduced to solve the SWEs in hydrodynamics with the free surface flows. For example, A space–time conservation method [1] differing from the traditional methods was applied successfully to solve the Saint Venant equations [2]. Fraccarollo and Toro [3] applied the weighted average flux method and Glaister [4,5] presented an approximate Riemann solver. Rao and Latha [6], and Nujic [7] used the modified Lax–Friedrich scheme, Savic and Holly [8] used the Godunov method, a scheme with numerical or artificial viscosity and/or adding a smoothing term to Finite Difference Equation (FDE) to dampen the oscillations [9-11].

A characteristic feature of all second- and higher order accurate schemes lies in producing dispersive errors near the vicinity of discontinuity/shocks [12]. These errors, which are manifest in the form of oscillations, need to be smoothed at the instant of their generation. Based on the above research results; the goal of the current work is to develop a mathematical model capable of dealing with hydraulic discontinuities such as steep fronts, hydraulic jump and drop, etc. The water governing equations has been solved by an implicit bi-diagonal numerical scheme based on the MacCormack's predictor-corrector technique. In the present work we have used the procedure originally suggested by Jameson et al. [13] to reduce numerical oscillations.

MATHEMATICAL MODEL

The conservation Laws of the governing flow equations for the physical domain, assuming that the flow is homogeneous, incompressible, 1D and viscous with hydrostatic pressure distribution, are [14]:

$$\frac{\partial \boldsymbol{E}}{\partial t}+\frac{\partial \boldsymbol{F}(\boldsymbol{E})}{\partial x}=\boldsymbol{Q}(\boldsymbol{E}) \quad (1)$$

in which the variables **E,F,** and **Q** are defined in matrix forms as follows:

$$\boldsymbol{E}=(h,q_x)^T \quad (2a)$$

$$\boldsymbol{F}(\boldsymbol{E})=\left(q_x,\frac{q_x^2}{h}+\frac{1}{2}gh^2\right)^T \quad (2b)$$

$$\boldsymbol{Q}(\boldsymbol{E})=\left[0,gh\left(s_{0x}-s_{fx}\right)\right]^T \quad (2c)$$

where q=component of discharge per unit width along x-direction; s_{ox}=bottom slope along x-direction,; and s_{fx} =friction slope along x-direction. The friction slopes s_{fx} is determined by the Manning formula

$$S_{fx}=\frac{n^2q_x^2}{h^{4/3}} \quad (3)$$

where n=Mannings's flow friction coefficient.

By utilizing a local characteristic approach, the equivalent nonconservative form of (1) can be expressed as

$$\frac{\partial \boldsymbol{E}}{\partial t}+\boldsymbol{A}\frac{\partial \boldsymbol{E}}{\partial x}=\boldsymbol{Q}(\boldsymbol{E}) \quad (4)$$

The Jacobian of the fluxes are,

$$\boldsymbol{A}=\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{E}}=\begin{bmatrix} 0 & 1 \\ c^2-\frac{q_x^2}{h^2} & 2\frac{q_x}{h} \end{bmatrix} \quad (5)$$

where c is celerity.$c^2=gh$.

The eigenvalues of matrices A is,

$$\boldsymbol{e}_A=\left(q_x/h+c,q_x/h\right)^T \quad (6)$$

The governing equations are known to be hyperbolic, which implies that A has complete set of independent and real eigenvector. Therefore, the Jacobian can be written in diagonalized form as[14]:

$$\boldsymbol{A}=\boldsymbol{e}\boldsymbol{e}_A\boldsymbol{e}^{-1} \quad (7)$$

where e and e^{-1}, are matrices and inverse matrices of eigenvectors of A.

NUMERICAL SCHEME

The scheme is an implicit finite-volume method for time-integrating the Navier-Stockes equations. It is second-order accurate in space and time, unconditionally stable, and very efficient in that no block or scalar tri-diagonal inversions need to be calculated. Eq.(4) is integrated by the following implicit predictor-corrector set of finite-difference equation[14,15]:

Predictor

$$\Delta \boldsymbol{E}_i^n = -\Delta t \left(\frac{\Delta^+ \boldsymbol{F}_i^n}{\Delta x} - \boldsymbol{Q}^n \right) \tag{8a}$$

$$\left(I - \Delta t \frac{\Delta^+ \boldsymbol{A}^n}{\Delta x} \right) \delta \boldsymbol{E}_i^{\overline{n+1}} = \Delta \boldsymbol{E}_i^n \tag{8b}$$

$$\boldsymbol{E}_i^{\overline{n+1}} = \boldsymbol{E}_i^n + \Delta \boldsymbol{E}_i^{\overline{n+1}} \tag{8c}$$

Corrector

$$\Delta \boldsymbol{E}_i^{\overline{n+1}} = -\Delta t \left(\frac{\Delta^- \boldsymbol{F}_i^{\overline{n+1}}}{\Delta x} - \boldsymbol{Q}^{\overline{n+1}} \right) \tag{9a}$$

$$\left(I + \Delta t \frac{\Delta^- \boldsymbol{A}^{\overline{n+1}}}{\Delta x} \right) \delta \boldsymbol{E}_i^{n+1} = \Delta \boldsymbol{E}_i^{\overline{n+1}} \tag{9b}$$

$$\boldsymbol{E}_i^{n+1} = 1/2 \left(\boldsymbol{E}_i^n + \boldsymbol{E}_i^{\overline{n+1}} + \delta \boldsymbol{E}_i^{n+1} \right) \tag{9c}$$

while $\Delta^+/\Delta x$ is one-sided forward difference; and$\Delta^-/\Delta x$ is one-sided backward difference. For example, the one-sided forward and backward differences in x direction for the flux **F** are

$$\frac{\Delta^+ \boldsymbol{F}_{i,j}}{\Delta x} = \frac{\boldsymbol{F}_{i+1,j} - \boldsymbol{F}_{ij}}{\Delta x}, \frac{\Delta^- \boldsymbol{F}_{i,j}}{\Delta x} = \frac{\boldsymbol{F}_{i,j} - \boldsymbol{F}_{i-1,ij}}{\Delta x} \tag{10a,b}$$

The equations can be solved by sweeping in the x-direction.

A characteristic feature of all second- and higher order accurate schemes lies in producing dispersive errors near the vicinity of discontinuity/shocks [12]. These errors, which are manifest in the form of oscillations, need to be smoothed at the instant of their generation. In the present work we have used the procedure originally suggested by Jameson et al. [13] to reduce numerical oscillations. As per this approach, the two flow variables are smoothed as

$$E_i^{n+1} = E_i^{n+1} + \xi_{i+1/2} \left(E_{i+1}^{n+1} - E_i^{n+1} \right) - \xi_{i-1/2} \left(E_i^{n+1} - E_{i-1}^{n+1} \right) \tag{11}$$

where

$$\xi_i = \frac{\left| h_{i+1}^{n+1} - 2h_i^{n+1} + h_{i-1}^{n+1} \right|}{\left| h_{i+1}^{n+1} \right| + 2\left| h_i^{n+1} \right| + \left| h_{i-1}^{n+1} \right|} \tag{12}$$

and

$$\xi_{i+1/2} = \mu \max \left(\xi_i, \xi_{i+1} \right) \tag{13}$$

The solution obtained by Eq. (11) is free from numerical oscillations. The smoothing mechanism, as can be seen by Eq. (12), is triggered only in oscillatory regions. For regions where the flow is uniform, the numerator in Eq. (12) goes to zero, leaving the solution computed using Eq. (9c) unaltered. The parameter μ in Eq. (13) is known as dissipation constant which controls the degree of smoothing. Based on trial and error, in this work we have selected its value to be 0.6.

INITIAL AND BOUNDARY CONDITIONS

At the initial time, still water is assumed inside the computation domain.

The boundaries are inflow boundary and outflow boundary. The inflow boundary condition is given as the water surface elevation or the discharge hydrograph per unit width at the upstream end; and the outflow boundary condition is given as the water surface elevation.

STABILITY CONDITION

The above-described numerical scheme is a time-marching method in which △t must be satisfied with Courant-Friedrichs-Levy Condition. For every point i of the computational domain the △t time step is expresses by $\Delta t = \min\left(DT\right)$,where

$$DT = \frac{\Delta x}{|u| + \sqrt{gh}}.$$

RESULTS AND DISCUSSION

Simulation of hydraulic jump and drop:

The hydraulic drop is basic physical phenomenon in natural rivers or open channel flows. Unlike a hydraulic jump, in which an abrupt increase of water surface occurs, it is characterized by a substantial decrease of water depth within a short distance along the flow direction when the flow changes from the subcritical to supercritical state. According to open channel hydraulics [16], both the hydraulic jump and the hydraulic drop will occur if the inflow is in the supercritical state in a fairly long channel, with a mild bed slope, followed by a fairly long channel with a steep bed slope. Such a complex flow forms a useful test problem.

The straight rectangular channel consists of two reaches with different slopes: an upstream horizontal reach (S=0), followed by a reach with a steep slope (S=0.03). The first reach is 14.5m long, and the second is 16.0m. Both are 1.4m wide (Figure1).

The entrance velocities and depth are u=3.571ms^{-1}, v=0ms^{-1} and h=0.06m, and the corresponding entrance Froude number is Fr=4.65; no exit boundary condition for depth is needed because there is a critical depth in the cross-section at the change in the slope, which automatically plays the part of the internal boundary condition for both hydraulic jump upstream and supercritical flow downstream.

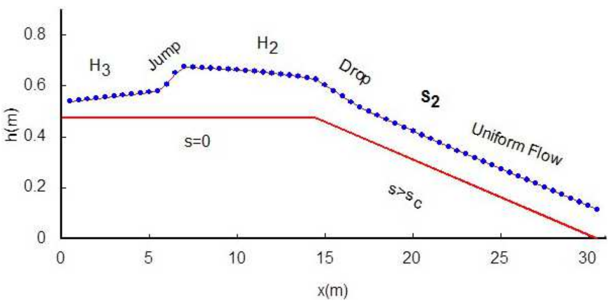

Fig.1. Profile of the depth in hydraulic jump and drop.

The following numerical parameters were used in the computations: △x=0.5m, △y=0.14m; Manning's coefficient n=0.019.

Figure 1 shows the profile of the central depth in the channel. As expected, the profile consists of a H3 curve, a hydraulic jump, a H2 curve, a hydraulic drop, a S2 curve and uniform flow.

Theoretically, a hydraulic jump will occur when the upstream Froude number Fr_1, depth h_1 and downstream depth h_2 satisfy the Bélanger formula $h_2/h_1 = \frac{1}{2}\left(\sqrt{8F_{r1}^2+1}-1\right)$. This can be used to check the results by the model. From the numerical results, the upstream Froude number, depth and downstream depth of the jump are Fr_1=1.715, h_1=0.111m and h_2=0.216m respectively. Substitution of Fr_1=1.715 and h_1=0.111m into the Bélanger formula results in the theoretical downstream depth required for the jump as $\hat{h}_2$ =0.219m. h_2 is thus very close to the theoretical downstream depth $\hat{h}_2$. The relative error is 1.43%.

In addition, the hydraulic drop occurs in the region around the change in slope and the central water depth (h=0.164m) at the change is very close to the theoretical critical depth (h_c=0.167m) calculated from the critical depth equation $h_c=\sqrt[3]{Q^2/gw^2}$ (where w is the width of the channel). After the hydraulic drop, the flow quickly approaches uniform flow along the second reach. The depth at the downstream end of the channel is h=0.111m, compared with a normal depth h_n=0.105 m calculated from the Manning formula for uniform flows [16].

Simulation of instantaneous opening of sluice gate problem:

The problem considered here is the total instantaneous opening of sluice gate on a flat and frictionless bed. This provides an ideal test case for shock-capturing schemes since analytical solution has been known. Figure 2 shows the illustration of the total instantaneous opening of sluice gate, where the initial upstream water depth is h_1 =10 m, and the downstream water depth is h_0 =5.The length of the computational region is 200m, and the sluice gate is located at *x*=200 m. The grid spacing is 1 m. The time step is 1 s. Figure 3 shows the water surface position, and Figure 4 shows the velocity distribution, 7.0s after the total instantaneous opening of sluice gate, where the solid line represents the analytical solution and the circle points illustrate the predicted results. It can be seen that the shallower the downstream water depth, then the faster the flood wave travels. The agreement between the analytical and numerical solutions is satisfactory.

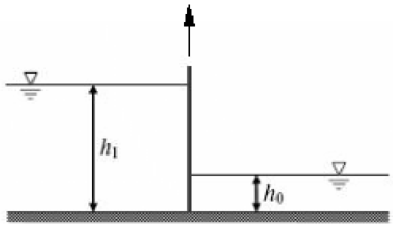

Fig.2 Illustration of opening of sluice gate

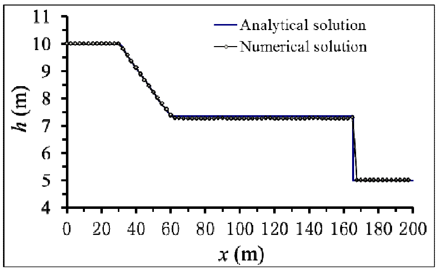

Fig. 3 Water surface position

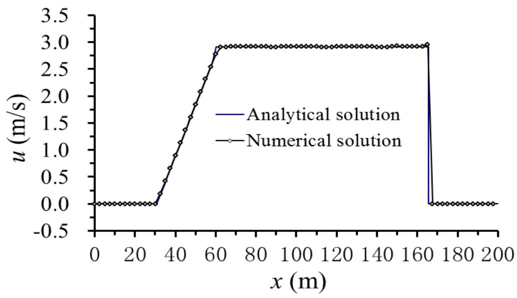

Fig. 4 Velocity distribution

CONCLUSION

The MacCormack scheme can effectively simulate the rapidly varying discontinuous water waves. The implicit bi-diagonal numerical scheme based on the MacCormack's predictor-corrector technique along with a proposed technique to reduce the oscillations near the vicinity of discontinuity/shocks has good convergence and stability when used to solve the discontinuous flow problems. The proposed mathematical model can effectively simulate the open channel flows from supercritical to subcritical state or from subcritical to supercritical state, and the 1D flood waves due to instantaneous opening of sluice gate or dam-break. The proposed model can be expanded to simulate 2D discontinuous flow problems.

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REFERENCES

[1] S.C . Chang, *J. Comput .Phys*., **1995**, (119),295–324.
[2] T. Molls, F . Molls, *J .Hydr .Engrg* ,ASCE., **1998**, (124),501–8.
[3] L . Fraccarollo, E.F .Toro, *J .Hydr .Res* ., **1995**, (33), 843–64.
[4] P. Glaister, *J .Hydr. Res*., **1988**, (26),293–306.
[5] P . Glaister, *Int J Numer Meth Fluids*, **1993**, (16),629–54.
[6] V.S. Rao and G. Latha, *Int J Numer Meth Fluids*, **1992** ,(14), 189–96.
[7] M .Nujic, *J .Hydr. Res*., **1995**, (3),101–11.
[8] L.J Savic and Jr .FM. Holly, *J .Hydr. Res*., **1993**, (1),187–204.
[9] T. Molls and M.H. Chauhdry, *J. Hydraul. Eng*. ASCE, **1995**,121 (6), 453–465.
[10] T. Molls and G. Zhao, *J. Hydraul. Eng*.,ASCE, **2000**,126 (6) , 437–444.
[11] A.J. Klonidis , J.V. Soulis, *J. Hydraul. Res*. **2001**,39 (4) , 393–402.
[12] SK Godunov, *Mat Sb* **1959**,(47),271-306.
[13] A Jameson, W Schmidt, Turkel .Numerical solutions of the Euler equations by finite volume methods using Runga-Kutta time stepping schemes. In: AIAA 14th Fluid and Plasma Dynamics Conference. Palo Alto, CA, AIAA, **1981**,1181-1259.
[14] W.L. Wei, Computational hydromechanics theory with applications, 1st Edition，,Xi'an，Shanxi Science and Technology Publishing House. **2010,92-93.**
[15] Y L Liu, YF Hong, WL Wei , *Journal of Chemical and Pharmaceutical Research*, **2014**, 6(4),265-269.
[16] R.J Fennema, M. H. Chandhry, *J. Hydr. Res*., **1989**,27(3), 321-331.