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Hilbert algebras in commutative BCK-algebras and ideal

Qiuna Zhang[1]*, Yan Yan[2], Huancheng Zhang[1] and Weixing Liu[1]

[1]Qinggong College, Hebei United University, Tangshan, China
[2]College of Science, Hebei United University, Tangshan, China

ABSTRACT

The notion of BCK-algebras was formulated first in 1966 by K. Iséki Japanese Mathematician. A BCK-algebra is an important class of logical algebras and was extensively investigated by several researchers. Here we well give the definition of Hilbert Algebras in commutative BCK-algebras and ideal.

Key words: Commutative BCK-algebra, Hilbert algebras, Ideal

INTRODUCTION

HILBERT ALGEBRAS IN COMMUTATIVE BCK-ALGEBRAS

Definition1.1 A Hilbert Algebras $(H,\to,1)$ in BCK-algebras is called commutative if and only if for any $x,y\in H$, it satisfies the following [1]:

$$(y\to x)\to x=(x\to y)\to y$$

Theorem1.2 For any Hilbert Algebras $(H,\to,1)$ in BCK-algebras，the following conditions are equivalent [2]:

（1） H is commutative，
（2） $(y\to x)\to x\le(x\to y)\to y$,
（3） $((x\to y)\to y)\to((y\to x)\to x)=1$

Theorem1.3 For any Hilbert Algebras $(H,\to,1)$ in BCK-algebras，the following conditions are equivalent:

（1）If $x\le z$ and $y\to z\le x\to z$， then $x\le y$,
（2）If $x,y\le z$ and $y\to z\le x\to z$， then $x\le y$,
（3）If $x\le y$ ， then $x=(x\to y)\to y$,
（4） H is commutative，
（5）If $y\to x=1$， then $((x\to y)\to y)\to x=1$

Proof: Obviously $(1)\Rightarrow(2)$.

$(2)\Rightarrow(3)$ because $(x\to y)\to y\le y$, $((x\to y)\to y)\to y\le x\to y$, adding (2) we hold $x\le(x\to y)\to y$, similarly $(x\to y)\to y\le x$, then $x=(x\to y)\to y$

$(3)\Rightarrow(4)$ Since $(x\to y)\to y\le x$, adding (3) we hold

$(x\to y)\to y=(((x\to y)\to y)\to x)\to x$ For $((y\to x)\to x)\to((x\to y)\to y)$

$=((y\to x)\to x)\to((((x\to y)\to y)\to x)\to x)=(((x\to y)\to y)\to x)\to(((y\to x)\to x)\to x)$

$=(((x\to y)\to y)\to x)\to(y\to x)\le y\to((x\to y)\to y)=1$

then $(y\to x)\to x\ge(x\to y)\to y$, by theorem1.2, H is commutative

$(4)\Rightarrow(1)$ If $x\le z$ and $y\to z\le x\to z$, then $z\to x=1$, $(x\to z)\to(y\to z)=1$ by (4) we hold

$y\to x=y\to((z\to x)\to x)=y\to((x\to z)\to z)=(x\to z)\to(y\to z)=1$ then $x\le y$

$(3)\Rightarrow(5)$ Obviously $(3)\Leftrightarrow(5)$

Theorem1.4 A Hilbert Algebras $(H,\to,1)$ in BCK-algebras is called commutative if and only if for any $x,y\in H$, it satisfies the following [3]:

$(y\to x)\to x=(((y\to x)\to x)\to y)\to y$

Proof: Necessity: because H is commutative, so $(y\to x)\to x=(x\to y)\to y$, by theorem1.3 (3) $x=(x\to y)\to y$, then $(y\to x)\to x=(((x\to y)\to y)\to y)\to y$, for $(y\to x)\to x=(x\to y)\to y$, then $(y\to x)\to x=(((y\to x)\to x)\to y)\to y$

Sufficiency: Suppose $(y\to x)\to x=(((y\to x)\to x)\to y)\to y$, $x\le y$ then $x=1\to x=(y\to x)\to x$ $=(((y\to x)\to x)\to y)\to y$ $=(x\to y)\to y$ by theorem1.3 (3) H is commutative.

Theorem 1.5 A Hilbert Algebras $(H,\to,1)$ in BCK-algebras is called commutative if and only if for any $x,y\in H$, it satisfies the following [4]:

(1) $(y\to x)\to x=(x\to y)\to y$,
(2) $z\to(y\to x)=y\to(z\to x)$,
(3) $x\to x=1$,
(4) $1\to x=x$

Proof: Necessity is obviously.

Sufficiency: first proof $(H,\to,1)$ is a Hilbert Algebras in BCK-algebras.

by $z\to(y\to x)=y\to(z\to x)$ we hold $y\to((y\to x)\to x)=(y\to x)\to(y\to x)=1$ BCI-2 holds.

If $y\to x=y\to x=1$, by (1), (4) then $x=1\to x=(y\to x)\to x=(x\to y)\to y=1\to y=y$, BCI-4 holds.

by (1) (2) then

$(z \to x) \to (y \to x) = y \to ((z \to x) \to x) = y \to ((x \to z) \to z) = (x \to z) \to (y \to z)$

so $(z \to x) \to (y \to x) = (x \to z) \to (y \to z)$ （*）

suppose $x = y$, $z = 1$, then $x \to 1 = (1 \to x) \to (x \to x) = (x \to 1) \to (x \to 1) = 1$

BCK-5 holds.

for（*）（4）and BCK-5，thus

$(y \to z) \to ((z \to x) \to (y \to x))$

$= (1 \to (y \to z)) \to ((x \to z) \to (y \to z))$

$= ((y \to z) \to 1) \to ((x \to z) \to 1)$

$= 1 \to 1$

$= 1$

BCI-1 holds.

（3） $x \to x = 1$ is BCI-3

thus， $(H, \to, 1)$ is a Hilbert Algebras in BCK-algebras.

finally，by（1） $(y \to x) \to x = (x \to y) \to y$,

So the Hilbert Algebras $(H, \to, 1)$ in BCK-algebras is commutative.

COMMUTATIVE IDEAL

Definition 2.1 A non empty subset H_0 of H is called commutative ideal of Hilbert Algebras $(H, \to, 1)$ in BCK-algebras，if it satisfies [5]：

（1） $1 \in H_0$,

（2）If $z \to (y \to x) \in H_0$, and $z \in H_0$, then for any $x, y, z \in H_0$ it satisfies the following

$((x \to y) \to y) \to x \in H_0$

Obviously， H is a commutative ideal of H ，we call it trivial commutative ideal.

Example 2.2 Let $H = \{1,2,3,4\}$, the ordinary operation $\to$ is given by：

$\to$	1	2	3	4
1	1	2	1	4
2	1	1	3	4
3	1	2	1	4
4	1	1	3	1

then $(H, \to, 1)$ is a Hilbert Algebras in BCK-algebras， $\{1,2\}$ is an ideal，but it not a commutative ideal

Theorem 2.3 A commutative ideal of a Hilbert Algebras in BCK-algebras must be an ideal，but the inverse is does not hold.

Proof：Suppose H_0 is a commutative ideal，if $y \to x \in H_0$ and $y \in H_0$ ，then

$y \to (1 \to x) \in H_0$, $y \in H_0$, by $((x \to y) \to y) \to x \in H_0$ we hold

$x = ((x \to 1) \to 1) \to x \in H_0$, so H_0 is an ideal From Example 2.2 ideal is not a commutative ideal.

Theorem 2.4 If H_0 is a commutative ideal if and only if for $y \to x \in H_0$ ，the following hold:

$((x \to y) \to y) \to x \in H_0$

proof：Necessity. If H_0 is a commutative ideal， $y \to x \in H_0$ ，then $1 \to (y \to x) \in H_0$, $1 \in H_0$

By the definition of commutative ideal $((x \to y) \to y) \to x \in H_0$ holds Sufficiency: Suppose H_0 is an ideal, and it satisfies $((x \to y) \to y) \to x \in H_0$, if $z \to (y \to x) \in H_0$, $z \in H_0$, by the definition of ideal $y \to x \in H_0$ holds, by $((x \to y) \to y) \to x \in H_0$, thus H_0 is a commutative ideal.

Theorem 2.5 Suppose H_0 is an ideal of a Hilbert Algebras in BCK-algebras and $x \in H_0$ if $y \le x$ then $y \in H_0$.
Proof: $y \le x$ implies $x \to y = 1 \in H_0$. Combining $x \in H_0$ and the definition of ideal of a Hilbert Algebras in BCK-algebras we obtain $y \in H_0$.

Theorem 2.6 For any Hilbert Algebras an implicative ideal must be positive implicative ideal but the inverse does not hold.

Proof: Suppose H_0 is an implicative ideal and $z \to (y \to x) \in H_0$ $z \to y \in H_0$
Since $(z \to y) \to (z \to (z \to x)) \le y \to (z \to x) = z \to (y \to x) \in H_0$
By Theorem 2.5 we get $(z \to y) \to (z \to (z \to x)) \in H_0$.
Combining $z \to y \in H_0$ and making use of H_0 is an ideal we have
$z \to (z \to x) \in H_0$
As $((z \to x) \to x) \to (z \to x) = z \to (((z \to x) \to x) \to x) = z \to (z \to x) \in H_0$
It follows that $1 \to (((z \to x) \to x) \to (z \to x)) \in H_0$ combining $1 \in H_0$ we obtain $z \to x \in H_0$ this means that H_0 is a positive implicative ideal.

Theorem 2.7 For any Hilbert Algebras an ideal H_0 is implicative if and only if for all $x, y \in H_0$ if $(x \to y) \to x \in H_0$ implies $x \in H_0$.

Proof: Sufficiency: Suppose H_0 is an ideal. If $z \to ((x \to y) \to x) \in H_0$ and $z \in H_0$.
By the definition of ideal we have $(x \to y) \to x \in H_0$ it follows that $x \in H_0$.

Necessity is evident.

Theorem 2.8 Theorem2.2 Suppose H_0 is a nonempty subset of Hilbert algebras in positive implicative BCK-algebras H then the following conditions are equivalent:

(1) H_0 is an ideal of Hilbert algebras in positive implicative BCK-algebras;
(2) H_0 is an ideal and for any x, y in H $y \to (y \to x) \in H_0$ implies $y \to x \in H_0$;
(3) H_0 is an ideal and for any x, y, z in H
$z \to (y \to x) \in H_0$ implie $(z \to y) \to (z \to x) \in H_0$;
(4) $1 \in H_0$ and $z \to (y \to (y \to x)) \in H_0, z \in H_0$ imply $y \to x \in H_0$.

Proof. $(1) \Rightarrow (2)$ If H_0 is an ideal of Hilbert algebras in positive implicative BCK-algebras by H_0 is an ideal. Suppose $y \to (y \to x) \in H_0$ since $y \to y = 1 \in H_0$
by definition $y \to x \in H_0$ 2holds.

$(2) \Rightarrow (3)$ Assume(2)and $z \to (y \to x) \in H_0$
$z \to (z \to ((z \to y) \to x))$
$= z \to ((z \to y) \to (z \to x)) = z \to (z \to (y \to x))$
$= (z \to z) \to (z \to (y \to x)) = 1 \to (z \to (y \to x)) = z \to (y \to x) \in H_0$

it follows that $z \to (z \to ((z \to y) \to x)) \in H_0$ by(2) $z \to ((z \to y) \to x) \in H_0$.
As $(z \to y) \to (z \to x) = z \to ((z \to y) \to x)$ then $(z \to y) \to (z \to x) \in H_0$ .which is(3).
$(3) \Rightarrow (4)$ It's clear that $1 \in H_0$. If $z \to (y \to (y \to x)) \in H_0, z \in H_0$ then
$y \to (y \to (z \to x)) = y \to (z \to (y \to x)) = z \to (y \to (y \to x)) \in H_0$
$z \to (y \to x) = y \to (z \to x) = 1 \to (y \to (z \to x)) = (y \to y) \to (y \to (z \to x))$
$= y \to (y \to (z \to x)) \in H_0$

since H_0 is an ideal and $z \in H_0$ thus $y \to x \in H_0$.(4)holds.
$(4) \Rightarrow (1)$ First proof H_0 is an ideal. Suppose $y \to x \in H_0$ and $y \in H_0$ then $y \to (1 \to (1 \to x)) \in H_0$ and $y \in H_0$
By(4) $1 \to x = x \in H_0$ H_0 is an ideal. Next let $z \to (y \to x) \in H_0$ and $z \to y \in H_0$
$(z \to y) \to (z \to (z \to x)) = z \to (y \to (z \to x)) = y \to (z \to (z \to x))$
$= y \to ((z \to z) \to (z \to x)) = y \to (1 \to (z \to x)) = y \to (z \to x)$
$= z \to (y \to x) \in H_0$ then $(z \to y) \to (z \to (z \to x)) \in H_0$.

Combining $z \to y \in H_0$ and using(4) $z \to x \in H_0$. This have proofed that H_0 is an ideal of Hilbert algebras in positive implicative BCK-algebras.

Theorem 2.9 Let $(H, \to, 1)$ is a Hilbert Algebras in BCK-algebras， then the nonempty subset H_0 of H is an implicative ideal if and only if it is both a commutative ideal and positive implicative ideal.

Proof: Suppose H_0 is an implicative ideal，by Theorem 2.6，an implicative ideal must be positive implicative ideal now we proof H_0 is also commutative.
To do this Let $y \to x \in H_0$， Since $((x \to y) \to y) \to x \le x$,
we have $x \to y \le (((x \to y) \to y) \to x) \to y$
Denote $u = ((x \to y) \to y) \to x$,
we obtain $(u \to y) \to u = ((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x)$
$\le (x \to y) \to (((x \to y) \to y) \to x) = ((x \to y) \to y) \to ((x \to y) \to x)$
$\le y \to x \in H_0$
Hence， $(u \to y) \to u \in H_0$ Making use of Theorem 2.7we get $u \in H_0$， so $((x \to y) \to y) \to x \in H_0$.
This proof that if $y \to x \in H_0$ implies $((x \to y) \to y) \to x \in H_0$， Therefore H_0 is a commutative ideal.
Sufficiency: Suppose H_0 is both a commutative ideal and positive implicative ideal，Let $(x \to y) \to x \in H_0$,
Since $(x \to y) \to ((x \to y) \to y) \le (x \to y) \to x$, $(x \to y) \to ((x \to y) \to y) \in H_0$ ， Making use of Theorem 2.8（1）we hold $(x \to y) \to y \in H_0$

Moreover，since $y \to x \le (x \to y) \to x$ ，we obtain $y \to x \in H_0$
By Theorem 2.4 we hold $((x \to y) \to y) \to x \in H_0$，combining $(x \to y) \to y \in H_0$ we obtain $x \in H_0$
Therefore H_0 is an implicative ideal.

Theorem 2.10 Suppose H_1 and H_2 are ideals of H let $H_1 \subseteq H_2$，if H_1 is commutative then so is H_2.

Proof：Let $y \to x \in H_2$，denote $u = y \to x$，then $y \to (u \to x) = u \to (y \to x) = 1 \in H_1$
Using the commutative of H_1 and Theorem 2.4，we have $(((u \to x) \to y) \to y) \to (u \to x) \in H_1$
because $H_1 \subseteq H_2$，then $u \to ((u \to x)((\to y) \to y) \to x) = (((u \to x) \to y) \to y) \to (u \to x) \in H_1$
Combining $u \in H_2$，we have $(((u \to x) \to y) \to y) \to x \in H_2$ As

$$((((u \to x) \to y) \to y) \to x) \to (((x \to y) \to y) \to x)$$
$$\le ((x \to y) \to y) \to (((u \to x) \to y) \to y)$$
$$\le ((u \to x) \to y) \to (x \to y)$$
$$\le x \to (u \to x)$$
$$= u \to 1 \in H_2$$

We obtain $((x \to y) \to y) \to x \in H_2$，thus H_2 is a commutative ideal.

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