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Generalized B-spline curve surface and its properties

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ABSTRACT

Generalized B-spline curve has achieved the transition from the uniform B-spline curve to the trigonometric polynomial B-spline curve, it encompasses uniform B-spline curves and trigonometric polynomial B-spline curves and countless curves between them. Constructed by the nature of the base functions to derive the definition of the curve and its properties, and then simply introduce the nature of the surface.

Keywords: Trigonometric polynomial, B-spline, basis functions, properties

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INTRODUCTION

Spline function is said the basic tools in computational geometry and approaching the geometry, and uniform B-spline curve is an important tool for geometric design curve modeling is widely used in computer aided geometric design [1].In 1962,Bézier[2-4] proposed a new method by changing the control polygon to design curve. In 1972, Forrest [5] pointed out that the new method to design curve is exactly designed Bernstein corresponding linear combination of the product of the control vertices, and the curve is a Bézier curve, which has many unique properties, such as the convex hull property, elevation property, symmetry, reduced variation, geometrical invariability and affine invariability, and so on. As promotion of the Bézier curve, we got a spline curve [6]. But spline curve can't accurately represent elliptic, spiral and some other algebraic curves. Based on the above shortcomings, then there will be a trigonometric polynomial B-spline curve, it can accurately represent elliptic and some other algebra curves [7]. However, for a given control point, the position of the B-spline curve is fixed, while the uniform B-spline curves , trigonometric polynomial B-spline curves and the vicinal spline curves require appropriate adjustment and select control points, which bring to the inconvenience. In order to avoid the phenomenon which by changing the shape of the curve polygon to adjust the shape of the curve, the weighting factor of the rational Bézier curves and rational B-spline curves can realize the adjustment of the curve [8]. So the unified representation of the B-spline curves, trigonometric polynomial B-spline curves and the vicinal spline curves is very necessary. In literature [9] ,the constructed ωB spline by adjusting the size of ω , α to achieve the unity of the B-spline curves, trigonometric polynomial B-spline curves and hyperbolic polynomial B-spline curves, but this method is complex, and the selection of parameter ω depend on the parameter α . Comprehensive the above ideas, we can achieve the transition from the uniform B-spline basis function to the trigonometric polynomial B-spline basis function by constructing a set of basis functions with shape parameters. In literature [10] ,the parameters λ and α of the constructed generalized trigonometric polynomial uniform B-spline curve are independent, it can change the shape of the curve by changing their values, and then the curves between uniform B-spline curves and trigonometric polynomial uniform B-spline curves can be obtained.

Then the research on the properties of the generalized B-spline curves and surfaces is very necessary, and this is the basis of this kind of curves and surfaces modeling which will be researched.

2. The basic knowledge

2.1 The structure of the basis functions

B-spline is defined in different ways, the more convenient in application is de Boor-Cox recursion formula, and the recursion formula is used to determine the basis functions of B-spline. So, for the generalized trigonometric polynomial uniform B-spline curve is also given by recursive basis function method, defined as follows:

$$N_{0,2}(t)=\begin{cases}\dfrac{\alpha(1+\lambda)}{2(1-\cos\alpha)}\sin(\alpha t)-\lambda t & 0\le t<1\\ \dfrac{\alpha(1+\lambda)}{2(1-\cos\alpha)}\sin\left[\alpha(2-t)\right]-\lambda(2-t) & 1\le t<2\\ 0 & else\end{cases}$$

$$N_{0,k}(t)=\int_{t-1}^{t}N_{0,k-1}(x)dx$$

$$N_{i,k}(t)=N_{0,k}(t-i)$$

Among them $\alpha\in(0,\pi]$, $\lambda\in[-1,0]$, $i=0,\pm1,\pm2,\cdots\pm n$, $k\ge2$.

Then $N_{i,k}(t)$ is called $k(k\ge2)$ orders generalized trigonometric polynomial uniform B-spline basis function, λ and α are called the two shape parameters of the basis function.

2.2 The nature of the basis functions

1、Non-negative: $N_{i,k}(t)\ge0$, $t\in R$;

2、Unit decomposable: $\sum N_{i,k}(t)\equiv1$;

3、Symmetry: $N_{i,k}(i+k-t)=N_{i,k}(t+i)$, $t\in[0,k]$;

4、Continuity: $N_{i,k}(t)$ are C^{k-2} orders continuous;

5、Local support: $N_{i,k}(t)=\begin{cases}>0 & t\in(i,i+k)\\ =0 & else\end{cases}$;

6、Derivation formula: $N_{i,k}'(t)=N_{i,k-1}(t)-N_{i+1,k-1}(t)$;

7、Linear independence: $\{N_{i,k}(t)\}$ are linearly independent in interval $(-\infty,+\infty)$, specifically, $N_{i,k}(t)$, $N_{i+1,k}(t)$, ⋯ ⋯, $N_{i+k,k}(t)$ $(n\ge k)$ are linearly independent in interval $[i+k-1,i+k,\cdots,i+n+1]$;

8、Degenerative: When $\lambda=0$, $N_{i,k}(t)$ are $k(k\ge2)$ orders trigonometric polynomial uniform B-spline basis functions,

When $\lambda=-1$, $N_{i,k}(t)$ are $k(k\ge2)$ orders uniform B-spline basis functions.

Obviously, the nature of 8 shows that the above definition of basis functions link triangular polynomial spline basis functions with uniform B-spline basis functions well via parameter λ, that is to say the curves and surfaces structured by this basis function encompasses all the curves and surfaces.

3 Generalized B-spline curve and its property

According to the definition of the Bézier curve generalized trigonometric polynomial uniform B-spline curve can be similarly defined as follows: Given n spatial vectors $P_i\in R^3$ or R^2 $(i=1,2,\cdots,n)$, then call the parametric curve segment

$$p_k(t)=\sum_{i=1}^{n}P_iN_{i,k}(t),\quad k\le t\le n+1,\quad n\ge k$$

is k orders generalized B-spline curve.

According to the properties of the basis function, we can derive the following properties of the curve:

Property 1 (geometrical invariability): Affine transformation was carried out on the curve, namely using linear transformation M and translation c to get the new curve:

$$\begin{aligned}
p_k^*(t) &= Mp_k(t)+c \\
&= M\sum_{i=1}^{n} P_i N_{i,k}(t) + c\sum_{i=1}^{n} N_{i,k}(t) \\
&= \sum_{i=1}^{n} MP_i N_{i,k}(t) + \sum_{i=1}^{n} cN_{i,k}(t) \\
&= \sum_{i=1}^{n}(MP_i + c)N_{i,k}(t) = \sum_{i=1}^{n} P_i^* N_{i,k}(t)
\end{aligned}$$

Thus it can be seen, $p_k^*(t)$ is the generalized trigonometric polynomial uniform B-spline curves corresponded by the new control points P_i^* which are obtained by the control points P_i of the original curve $p_k(t)$ corresponding affine transform. After the original curve control points corresponding affine transform new control vertices corresponding generalized trigonometric polynomial uniform B-spline curves. This also explains the generalized trigonometric polynomial uniform B-spline curve does not depend on the selection of selected coordinate system, and has geometrical invariability.

Property 2 (convex hull): Curve $p_k(t)$ is located within the control polygon which is composed by vertices P_i.

Property 3 (derived function):

$$\begin{aligned}
p_k'(t) &= \sum_{i=1}^{n} P_i N_{i,k}'(t) \\
&= \sum_{i=1}^{n} P_i\left[N_{i,k-1}(t) - N_{i+1,k-1}(t)\right] \\
&= p_{k-1}(t) - p_{k-1}(t-1)
\end{aligned}$$

Property 4 (symmetry): If we reverse the order of the control vertices P_i of the curve $p_k(t)$, make $P_i^* = P_{n-i}\ (i = 1,2,\cdots,n)$, the new generalized trigonometric polynomial uniform B-spline curve is:

$$\begin{aligned}
p_k^*(t) &= \sum_{i=1}^{n} P_i^* N_{i,k}(t) = \sum_{i=1}^{n} P_{n-i} N_{i,k}(t) \\
&= \sum_{i=1}^{n} P_i N_{n-i,k}(t) = \sum_{i=1}^{n} P_i N_{i,k}(t-n+2i) \\
&= \sum_{i=1}^{n} P_i N_{i,k}(n+k-t) = p_k(n+k-t) \\
&= p_k(n+k-t)
\end{aligned}$$

If $p_k^*(t)$ and $p_k(t)$ express the same curve, then it requires only one parameter change $s = n+k-t$, and the direction of the curve is opposite to the original.

Property 5 (locality): According to the local support of $N_{i,k}(t)$, namely: $N_{i,k}(t) = \begin{cases} >0 & t \in (i, i+k) \\ =0 & else \end{cases}$, the curve $p_k(t)$ is mainly distributed in the interval $(i, i+k)$.

Property 6 (approximation): For the two parameters λ and α of the base function, Their scope respectively are $-1 \le \lambda \le 0$ and $0 < \alpha \le \pi$ 。But in fact, with the increase of order, for the arbitrary parameter α, the value range of the parameter λ can be enlarged; the value of the parameter λ is no longer limited in the interval $[-1,0]$. That is to say, when the value of the parameter α is fixed, the value of the parameter λ is smaller, the curve is closer to the control polygon, as shown in Figure 1: $\alpha = 1.5$, the value of the parameter λ respectively are $\lambda = 3$ （strip line）, $\lambda = 1$（full line）and $\lambda = -1$（dotted line）. Obviously, When α is unchanged, the parameter λ smaller, the curve is closer to the control polygon.

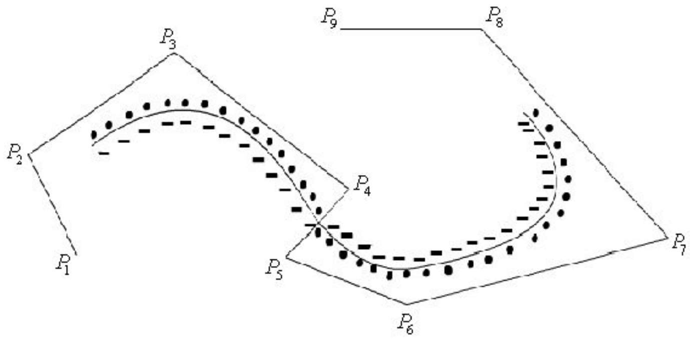

Figure 1 When α is unchanged, the shape of the curve with the parameter λ

And when the value of the parameter λ is fixed, the value of the parameter α is smaller, the curve is closer to the control polygon, as shown in Figure 2: λ=2 , the value of the parameter α respectively are $\alpha = 2.5$ （dotted line）, $\alpha = 1.5$ （full line） and $\alpha = 0.5$ （square line）.

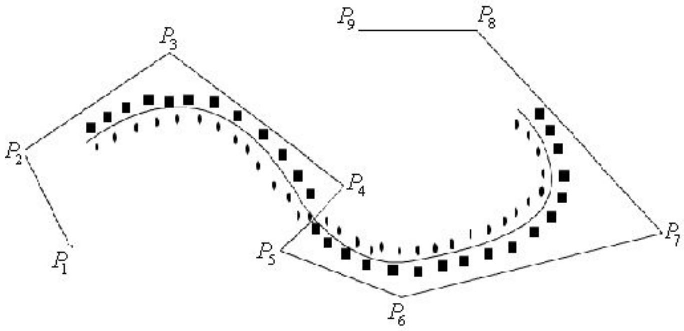

Figure 2 When λ is unchanged, the shape of the curve with the parameter α

Further, the closed curve also have similar properties, that is to say, when the value of the parameter α is fixed, the value of the parameter λ is smaller, the curve is closer to the control polygon, as shown in Figure 3: $\alpha = \frac{\pi}{2}$, curve 1,2,3,4 respectively express the shape of the curve when the value of the parameter λ is $\lambda = 2$, $\lambda = 1$, $\lambda = 0.5$ and $\lambda = 0$.

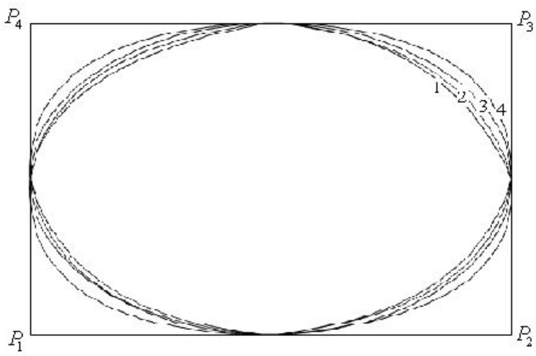

Figure 3 the changes of the closed curve with the different value of λ

4 Generalized B-spline surfaces

In the second part of this paper, the structured basis function, with vector as a factor, we can construct the space parametric curve, and it is the generalized trigonometric polynomial uniform B-spline curve in the third part of this paper. In order to construct the polynomial parametric surface, we need binary polynomial basis functions. Generally, there are two ways to the promote functions from one dollar to multivariate function: one way is using the tensor product (product) to promote; another way is using simplex to promote. According to the construction method of Bézier surface, this part will introduce the tensor product type of generalized trigonometric polynomial uniform B-spline surfaces on a rectangular domain (unit rectangular domain).

Firstly, the tensor product type of generalized trigonometric polynomial uniform B-spline basis function defined as follows:

$$N_{i,j}^{m,n}(u,v) = N_i^m(u) N_j^n(v) \qquad i = 1,2,\cdots,m \qquad j = 1,2,\cdots,n$$

Obviously, the basic properties of the dollar basis function can directly be extended to the tensor product type of generalized B-spline basis function $N_{i,j}^{m,n}(u,v)$. For example:

1、Non-negative: $N_{i,j}^{m,n}(u,v) \ge 0$, $(u,v) \in [0,1]\times[0,1]$;

2、Local support: The local support of binary generalized B -spline basis function $N_{i,j}^{m,n}(u,v)$ is the tensor product of $N_i^m(u)$ and $N_j^n(v)$, namely:

$$N_{i,j}^{m,n}(u,v) = \begin{cases} > 0 & (u,v) \in (i,i+m)\times(j,j+n) \\ = 0 & else \end{cases};$$

3、Unit degradation: $\sum_{i=1}^{m}\sum_{j=1}^{n} N_{i,j}^{m,n}(u,v) \equiv 1$;

4、Derivative function:

$$\frac{\partial}{\partial u} N_{i,j}^{m,n}(u,v) = N_{i,m}{}'(u) N_{j,n}(v) = N_{j,n}(v)\left[N_{i,m-1}(u) - N_{i+1,m-1}(u)\right]$$
$$= N_{i,j}^{m-1,n}(u,v) - N_{i+1,j}^{m-1,n}(u,v),$$
$$\frac{\partial}{\partial v} N_{i,j}^{m,n}(u,v) = N_{j,n}{}'(v) N_{i,m}(u) = N_{i,m}(u)\left[N_{j,n-1}(v) - N_{j+1,n-1}(v)\right]$$
$$= N_{i,j}^{m,n-1}(u,v) - N_{i,j+1}^{m,n-1}(u,v);$$

5、Degenerative: As $\lambda = 0$, $N_{i,j}^{m,n}(u,v)$ is $k(k \ge 2)$ orders tensor product of the trigonometric polynomial uniform B-spline basis functions,

As $\lambda = -1$, $N_{i,j}^{m,n}(u,v)$ is $k(k \ge 2)$ orders tensor product of the uniform B-spline basis functions;

Definition: Given spatial vectors $P_{i,j} \in R^3$ $(i = 1,2,\cdots,m, j = 1,2,\cdots,n)$, $N_i^m(u)$ and $N_j^n(v)$ are the defined generalized trigonometric polynomial uniform B-spline basis functions, among them $(u,v) \in [0,1]\times[0,1]$, surface defined as follows:

$$P(u,v) = \sum_{i=1}^{m}\sum_{j=1}^{n} P_{i,j} N_{i,j}^{m,n}(u,v) = \sum_{i=1}^{m}\sum_{j=1}^{n} P_{i,j} N_i^m(u) N_j^n(v)$$

is called generalized trigonometric polynomial uniform B-spline surface, $P_{i,j}$ are called control vertices. Connecting the two adjacent control vertices which are in the same line and column by a straight line segment successively, we can obtained a control grid of the surface, as shown in Figure 4:as $\lambda = -1$, generalized B-spline surface is uniform B-spline surface.

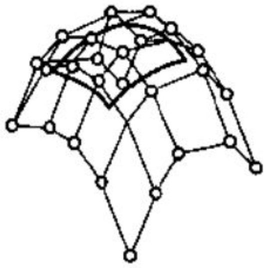

Figure 4 Uniform B-spline surfaces and its control grid

As a promotion of the generalized trigonometric polynomial uniform B-spline curves, the tensor product of generalized surfaces have some properties similar to that of the curve.

Property 1 (geometrical invariability): According to the unit degradation in this part, we can get the conclusion.

Property 2 (property of convex hull): According to the non-negative and the unit degradation, we know that the tensor product of generalized B-spline surfaces within the convex hull of the control grid.

Property 3 (property of slice): generalized B-spline surfaces in each cell $\left[u_i,u_{i+1}\right]\times\left[v_i,v_{i+1}\right]$ are $p\times q$ orders polynomial surfaces about parameter u and v, namely:

$$P\left(u,v\right)=\sum_{k=i-p}^{i}\sum_{l=j-q}^{j}P_{k,l}N_{k,l}^{p,q}\left(u,v\right).$$

Therefore, the tensor product of generalized B-spline surface is a piecewise polynomial surface. As shown in Figure 5:

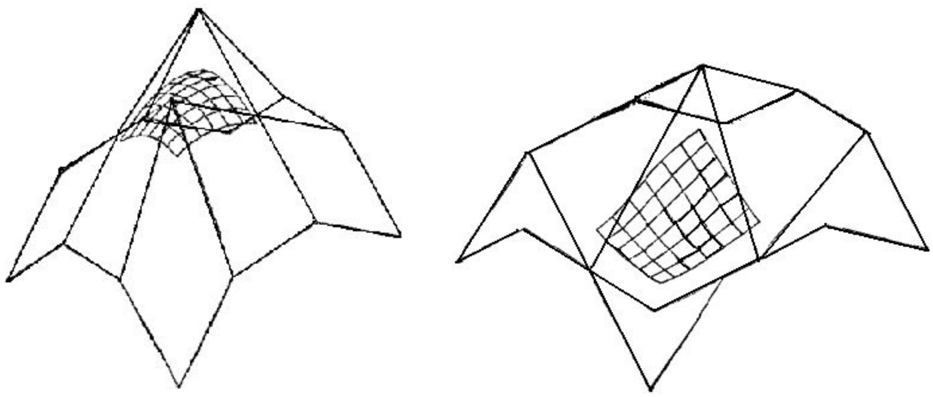

Figure 5 Uniform B-spline surfaces

Property 4 (property of isoparms): Suppose $v=v^*$ and $u=u^*$, then there will be:

$$P\left(u,v^*\right)=\sum_{i=0}^{m}\left(\sum_{j=0}^{n}P_{i,j}N_j^n\left(v^*\right)\right)N_i^m\left(u\right)$$

$$P\left(u^*,v\right)=\sum_{j=0}^{n}\left(\sum_{i=0}^{m}P_{i,j}N_i^m\left(u^*\right)\right)N_j^n\left(v\right)$$

Obviously, $v=v^*$ and $u=u^*$ respectively are m orders and n orders generalized B-spline curves, and the two curves are the tensor product of generalized B-spline curve's isoparms.

In addition, due to the degenerative of the basis function, the tensor product of generalized B-spline surface also can be divided into two kinds of special surfaces, that is the uniform B-spline surface(as $\lambda=-1$)and trigonometric polynomial uniform B-spline surface（as $\lambda=0$ ）.Naturally, they also have the geometric properties of these two kinds of surfaces, such as: property of multiple node continuous order, property of multiple node interpolation, property of degradation, etc.

CONCLUSION

The basis function with shape parameters which is defined in this paper, on the one hand, the curve structured in this way include the uniform B-spline curve, trigonometric polynomial B-spline curve and the curves between them ($k \geq 2$), on the other hand, under the condition of without changing the control polygon, we only need to change the parameters so that we can adjust or revise the shape and position of the curve or the surface. As can be seen from the properties of the curves and surfaces, the introduced shape parameters make curves more flexible, and enrich the curve and surface modeling. But how to change the shape parameters which makes the design of curve and surface to better meet the needs of the people and the actual has certain significance to research, and it remains to further research.

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