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Composition operator from weighted Bergman space to q-Bloch space in politics

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ABSTRACT

In the paper, we study the composition operator C_φ *from weighted Bergman space to q-Bloch space in politics, we obtain the sufficient and necessary condition for* C_φ *to be bounded or compacted operator from weighted Bergman space to q-Bloch space.*

Key words: Composition operator; weighted Bergman space; q-Bloch space ness; compactness

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INTRODUCTION

Let $D=\{z\in C:|z|<1\}$ be the open unit disk in C, and use

$$D^n = D\times D\times\cdots\times D=\{z=(z_1,z_2,\cdots,z_n)\in C^n:|z_k|<1,1\le k\le n\}$$

To denote the polydisk in C^n and use ∂D^n to denote the full topological boundary of D^n. Let $H(D^n)$ denote the space of all holomorphic functions in D^n.

For $0<p<+\infty$ and $\alpha>-1$ the weighted Bergman space $A_\alpha^p(D^n)$ consists of all functions $f\in H(D^n)$ such that

$$\|f\|_{\alpha,p}^p=\int_{D^n}|f(z)|^p dv_\alpha(z)<+\infty$$

where

$$dv_\alpha(z_1,\cdots,z_n)=dA_\alpha(z_1)\cdots dA_\alpha(z_n)=(\alpha+1)^n\prod_{k=1}^n(1-|z_k|^2)^\alpha dA(z_1)\cdots dA(z_n)$$

here

$$dA_\alpha(z)=(\alpha+1)^\alpha(1-|z|^2)^\alpha dA(z)$$

is a weighted area measure on D^n with $dA(z)$ being normalized Lebesgue area measure on D. When $1\le p<+\infty$, $A_\alpha^p(D^n)$ is a Banach space with the norm $\|\ \|_{\alpha,p}$. If $0<p<1$, the space $A_\alpha^p(D^n)$ is a complete metric space with the following distance: $\rho(f,g)=\|f-g\|_{\alpha,p}^p$.

For $q>0$, $f\in H(D^n)$ is said to belong to the q-Bloch space $B^q(D^n)$ provided that

$$\sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial f}{\partial z_k}(z)\right|<+\infty$$

It is well known that B^q is a Banach space under the norm:

$$\|f\|_{B^q}=f(0)+\sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial f}{\partial z_k}(z)\right|<+\infty$$

When $q=1, B^1=B$ is the classical Bloch space. Let $\varphi=(\varphi_1,\cdots,\varphi_n)$ be a holomorphic self-map of D^n. The composition operator C_φ is defined by $C_\varphi f=f\circ\varphi$, $f\in H(D^n)$.Composition operators acting on Bergman space and Bloch space have been well understood (see [1-5]). Recently several authors have studied composition operator on different spaces of analytic functions. When Bloch spaces ere characterized in [6]. Tang and Hu [7] have got the characterization of bounded or compact composition operators between weighted Bergman space and q-Bloch space on the unit disk D. For the higher-dimensional case, zhang [8] characterized the boundedness or compactness of the composition type operator from Bergman space to μ-Bloch type space in the unit ball. The main purpose of this paper is to discuss the conditions for which C_φ is a bounded operator or compact operator from weighted Bergman to q-Bloch space on the polydiscs.

Throughout the paper, C denotes a position constant, whose value may change from one occurrence to the next one.

1. **The boundedness of** C_φ.

First, we give the following useful Lemmas.

Lemma.2.1 Let $0<p<+\infty$ and $-1<\alpha<+\infty$, then

$$|f(z)|\le\frac{C\|f\|_{\alpha,p}}{\prod_{k=1}^{n}(1-|z_k|^2)^{\frac{2+\alpha}{p}}}$$

For all $f\in A_\alpha^p(D^n)$ and $z_k\in D^n$.

Proof let $\beta(z,w)$ denote the Bergman metric on D^n. For any $z\in D^n$ and $R>0$, we use

$$D(z,r)=\{w\in D^n;\beta(z,w)<R\}$$

For the Bergman metric ball at z with radius R. It is well known that for any fixed $R>0$, we have

$$v_\alpha(D(z,R)):\ \prod_{k=1}^{n}(1-|z_k|^2)^{2+\alpha}$$

Now, let any $f\in A_\alpha^p(D^n)$, then $f\in H(D^n)$ and $|f|^p$ is the subharmonic. By the sub-mean-value property for $|f|^p$, we have

$$|f(z)|^p\le\frac{C}{v_\alpha(D(z,R))}\int_{D(z,R)}|f(w)|^p dv_\alpha(w)$$

$$:\ \frac{C}{\prod_{k=1}^{n}(1-|z_k|^2)^{2+\alpha}}\int_{D^n}|f(w)|^p dv_\alpha(w)$$

$$= \frac{C\|f\|_{\alpha,p}}{\prod_{k=1}^{n}(1-|z_k|^2)^{2+\alpha}} \tag{2.1}$$

The result follows from (2.1).

Lemma.2.2 Suppose $0 < p < +\infty$ and $-1 < \alpha < +\infty$, then

$$f \in B^{\frac{2+\alpha+p}{p}} \quad \text{and} \quad \|f\|_{B^{\frac{2+\alpha+p}{p}}} \le C\|f\|_{\alpha,p}$$

for any $f \in A_\alpha^p(D^n)$.

Theorem2.1et $0 < p, q < +\infty$, $-1 < \alpha < +\infty$ and φ be a holomorphic self-map of D^n .Then $C_\varphi : A_\alpha^p(D^n) \to B^q(D^n)$ is a bounded composition operator if and only if the following is satisfied:

$$\sup_{z\in D^n}\sum_{k,l=1}^{n}\frac{(1-|z_k|^2)^q}{\prod_{l=1}^{n}(1-|\varphi_l(z)|^2)^{\frac{2+\alpha+p}{p}}}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right| < +\infty \tag{2.2}$$

Proof. Suppose that (2.2) holds. Suppose any positive constant M . Let

$$M = \sup_{z\in D^n}\sum_{k,l=1}^{n}\frac{(1-|z_k|^2)^q}{\prod_{l=1}^{n}(1-|\varphi_l(z)|^2)^{\frac{2+\alpha+p}{p}}}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right| < +\infty$$

For any $f \in A_\alpha^p(D^n)$, by Lemma 2.1 and Lemma 2.2 we have

$$\|C_\varphi f\|_{B^q} = |f(\varphi(0))| + \sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial(f\circ\varphi)}{\partial z_k}(z)\right|$$

Clearly,

$$|f(\varphi(0))| \le \frac{C\|f\|_{\alpha,p}}{\prod_{k=1}^{n}(1-|\varphi_k(0)|^2)^{\frac{2+\alpha}{p}}} \tag{2.3}$$

holds. And we have

$$\sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial(f\circ\varphi)}{\partial z_k}(z)\right|$$

$$\le \sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial f}{\partial w_l}(\varphi(z))\right|\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|$$

$$= \sup_{z\in D^n}\sum_{k,l=1}^{n}\frac{(1-|z_k|^2)^q}{\prod_{l=1}^{n}(1-|\varphi_l(z)|^2)^{\frac{2+\alpha+p}{p}}}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\left|\frac{\partial f}{\partial w_l}(\varphi(z))\right|\prod_{l=1}^{n}(1-|\varphi_l(z)|^2)^{\frac{2+\alpha+p}{p}}$$

$$\le \sup_{z\in D^n}\sum_{k,l=1}^{n}\frac{(1-|z_k|^2)^q}{\prod_{l=1}^{n}(1-|\varphi_l(z)|^2)^{\frac{2+\alpha+p}{p}}}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\sup_{z\in D^n}\sum_{l=1}^{n}\left|\frac{\partial f}{\partial w_l}(\varphi(z))\right|\prod_{l=1}^{n}(1-|\varphi_l(z)|^2)^{\frac{2+\alpha+p}{p}}$$

$$\leq \sup_{z\in D^n}\sum_{k,l=1}^{n}\frac{(1-|z_k|^2)^q}{\prod_{l=1}^{n}(1-|\varphi_l(z)|^2)^{\frac{2+\alpha+p}{p}}}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\sup_{z\in D^n}\sum_{l=1}^{n}\left|\frac{\partial f}{\partial w_l}(\varphi(z))\right|(1-|\varphi_l(z)|^2)^{\frac{2+\alpha+p}{p}}$$

$$\leq M\|f\|_{B^{\frac{2+\alpha+p}{p}}}\leq MC\|f\|_{\alpha,p} \qquad (2.4)$$

By (2.3) and (2.4), we can obtain that C_φ is a bounded composition operator from $A_\alpha^p(D^n)$ to $B^q(D^n)$.

Conversely, suppose C_φ is a bounded composition operator from $A_\alpha^p(D^n)$ to $B^q(D^n)$. Then we can easily obtain $\varphi_l\in B^q(D^n)$ and

$\sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|<+\infty$ by taking $f(z)=z_l\ (l=1,\ldots,n)$ in $A_\alpha^p(D^n)$,

respectively. In order to prove (2.2), for any $w\in D^n$, we take

$$f_w(z)=\prod_{l=1}^{n}\left[\frac{1-|w_l|^2}{(1-\overline{w_l}z_l)^2}\right]^{\frac{2+\alpha}{p}}$$

Where $w_l=\varphi_l(z)$, then $\|f\|_{\alpha,p}\leq C$. Here, we may fix some $l\ \ (l=1,\ldots,n)$ without loss of generality. Thus we have

$$C\|C_\varphi\|\geq\|C_\varphi\|\|f_w\|_{\alpha,p}\geq\|C_\varphi f_w\|_{B^q}$$

$$\geq\sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial f_w}{\partial w_l}(\varphi(z))\right|\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|$$

$$=\sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\left(\frac{2+\alpha}{p}\right)^n\prod_{l=1}^{n}\frac{1}{(1-|\varphi_l(z)|^2)^{\frac{2+\alpha}{p}-p}}\bullet\frac{2|\varphi_l(z)|}{(1-|\varphi_l(z)|^2)^2}$$

$$=\frac{2^n(2+\alpha)^n}{p^n}\sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\prod_{l=1}^{n}\frac{1}{(1-|\varphi_l(z)|^2)^{\frac{2+\alpha+p}{p}}}\bullet|\varphi_l(z)| \qquad (2.5)$$

For any $\delta\in(0,1)$, here we will discuss with the following two cases.

Case Ⅰ, if $|\varphi_l(z)|>\delta$, by (2.5) we have

$$\sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\prod_{l=1}^{n}\frac{1}{(1-|\varphi_l(z)|^2)^{\frac{2+\alpha+p}{p}}}<+\infty \qquad (2.6)$$

Case Ⅱ, if $|\varphi_l(z)|\leq\delta$, by $\varphi_l\in B^q(D^n)$ and $\sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|<+\infty$ we can obtain

$$\sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\prod_{l=1}^{n}\frac{1}{(1-|\varphi_l(z)|^2)^{\frac{2+\alpha+p}{p}}}$$

$$\leq\sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\left(\frac{1}{1-\delta^2}\right)^{\frac{n(2+\alpha+p)}{p}}<+\infty \qquad (2.7)$$

By (2.5), (2.6), (2.7) and l is random then (2.2) holds.

This ends the proof of Theorem 2.1.

2. The compactness of C_φ

Lemma.3.1 Let $0<p,q<+\infty$, $-1<\alpha<+\infty$ and φ be a holomorphic self-map of D^n.Then C_φ is a compact operator from $A_\alpha^p(D^n)$ to $B^q(D^n)$ if and on if for any bounded sequence $\{f_j\}_{j=1}^\infty$ in $A_\alpha^p(D^n)$ which converges to 0 uniformly on compact subset of D^n, we have $\|C_\varphi f_j\|_{B^q}\to 0$ as $j\to\infty$.

Proof. he result can be proved by using Montel theorem and the definition of the compact operator, the details are omitted here.

Theorem3.1. Let $0<p,q<+\infty$, $-1<\alpha<+\infty$ and φ be a holomorphic self-map of D^n.Then $C_\varphi : A_\alpha^p(D^n)\to B^q(D^n)$ is a compact composition operator if and only if the following are all satisfied:

[1] $\varphi_l \in B^q(D^n)$ for all $l\in\{1,\ldots,n\}$ (3.1)

[2] $$\lim_{|\varphi(z)|\to\partial D^n}\sup_{z\in D^n}\sum_{k,l=1}^{n}\frac{(1-|z_k|^2)^q}{\prod_{l=1}^{n}(1-|\varphi_l(z)|^2)^{\frac{2+\alpha+p}{p}}}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|=0 \quad (3.2)$$

Proof**.** Suppose that (3.1) and (3.2) hold. Then for any $\varepsilon>0$, there exists $0<\delta<1$ such that

$$\sup_{z\in D^n}\sum_{k,l=1}^{n}\frac{(1-|z_k|^2)^q}{\prod_{l=1}^{n}(1-|\varphi_l(z)|^2)^{\frac{2+\alpha+p}{p}}}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|<\varepsilon \quad (3.3)$$

As $|\varphi_l(z)|^2>1-\delta$.

Let $\{f_j\}$ be any a sequence $\{f_j\}$ in $A_\alpha^p(D^n)$ which converges to 0 on compact subset of D^n satisfying $\|f_j\|_{a,p}\le C$. Then $\{f_j\}$ and $\left\{\frac{\partial f_j}{\partial z_k}\right\}$ converges to 0 uniformly on $E=\{w:|w|^2\le 1-\delta\}$, where E is any a compact subset of D^n.

If $dist(\varphi(z),\partial D^n)<\delta$ then, from (3.3) and Lemma 2.2, we have

$$\sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial(f_j\circ\varphi)}{\partial z_k}(z)\right|$$

$$\le\sup_{z\in D^n}\sum_{k,l=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\left|\frac{\partial f_j}{\partial w_l}(\varphi(z))\right|$$

$$=\sup_{z\in D^n}\sum_{k,l=1}^{n}\frac{(1-|z_k|^2)^q}{\prod_{l=1}^{n}(1-|\varphi_l(z)|^2)^{1+\frac{2+\alpha}{p}}}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\left|\frac{\partial f_j}{\partial w_l}(\varphi(z))\right|\prod_{l=1}^{n}(1-|\varphi_l(z)|^2)^{1+\frac{2+\alpha}{p}}$$

$$\le \sup_{z\in D^n}\sum_{k,l=1}^{n}\frac{(1-|z_k|^2)^q}{\prod_{l=1}^{n}(1-|\varphi_l(z)|^2)^{1+\frac{2+\alpha}{p}}}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\sup_{z\in D^n}\sum_{l=1}^{n}\left|\frac{\partial f_j}{\partial w_l}(\varphi(z))\right|\prod_{l=1}^{n}(1-|\varphi_l(z)|^2)^{1+\frac{2+\alpha}{p}}$$

$$\le \varepsilon\sup_{z\in D^n}\sum_{l=1}^{n}(1-|\varphi_l(z)|^2)^{1+\frac{2+\alpha}{p}}\left|\frac{\partial f_j}{\partial w_l}(\varphi(z))\right| < \|f_j\|_{B^{\frac{2+\alpha+p}{p}}}\cdot\varepsilon \tag{3.4}$$

If $dist(\varphi(z),\partial D^n)\ge\delta$, and we assume that $\{f_j\}$ be any a sequence $\{f_j\}$ in $A_\alpha^p(D^n)$ which converges to 0 on compact subset of D^n satisfying $\|f_j\|_{a,p}\le C$. Then $\{f_j\}$ and $\left\{\frac{\partial f_j}{\partial z_k}\right\}$ converges to 0 uniformly on $E=\left\{w:|w|^2\le 1-\delta\right\}$. By condition (3.1), we have

$$\sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial(f_j\circ\varphi)}{\partial z_k}(z)\right|$$

$$\le \sup_{z\in D^n}\sum_{k,l=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial f_j}{\partial w_l}(\varphi(z))\right|\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|$$

$$\le \sup_{z\in D^n}\sum_{k,l=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\sup_{z\in D^n}\left|\frac{\partial f_j}{\partial w_l}(\varphi(z))\right|$$

$$\le \|\varphi_l\|_{B,q}\sup_{z\in D^n}\left|\frac{\partial f_j}{\partial w_l}(\varphi(z))\right|\to 0\ (j\to\infty)\quad . \tag{3.5}$$

We can prove easily $\left|f_j(\varphi(0))\right|\to 0\quad (j\to\infty)$,and by(3.4),(3.5)we have

$$\|C_\varphi f_j\|_{B,q}=\|f_j\circ\varphi\|_{B,q}\to 0\ (j\to\infty)\quad .$$

This means that C_φ is a compact operator from $A_\alpha^p(D^n)$ to $B^q(D^n)$.

Conversely, for any $l\in\{1,\cdots,n\}$, by taking $f(z)=z_l\in A_\alpha^p$, we have $(C_\varphi f)(z)=\varphi(z_l)\in B^q$, so condition (3.1) must hold.

Assume that condition (3.2) fails. Then there exists constant $\varepsilon_0>0$ and sequence $\{z^j\}\subset D^n$ satisfying $\varphi(z^j)\to\partial D^n$ as $j\to\infty$, such that

$$\sup_{z\in D^n}\sum_{k,l=1}^{n}\frac{(1-|z_k|^2)^q}{\prod_{l=1}^{n}(1-|\varphi_l(z)|^2)^{\frac{2+\alpha+p}{p}}}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\ge\varepsilon_0 \tag{3.6}$$

For any $w \in D^n$, we take

$$f_j(z)=\prod_{l=1}^{n}\left[\frac{1-|w_l|^2}{(1-\overline{w_l}z_l)^2}\right]^{\frac{2+\alpha}{p}}$$

Where $w_l=\varphi_l(z)$ then $\|f_j\|_{\alpha,p}=1$ and $\{f_j\}$ converges to 0 uniformly on compact subset of D^n. Because C_φ is a compact operator from $A^p_\alpha(D^n)$ to $B^q(D^n)$, we have

$$\|C_\varphi f_j\|_{B,q}=\|f_j\circ\varphi\|_{B,q}\to 0\ (j\to\infty) \qquad (3.7)$$

But from (3.6) we have

$$\|C_\varphi f_j\|_{B^q}\ge \sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial f_j}{\partial w_l}(\varphi(z))\right|\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|$$

$$=\sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\left(\frac{2+\alpha}{p}\right)^n\prod_{l=1}^{n}\frac{1}{(1-|\varphi_l(z)|^2)^{\frac{2+\alpha}{p}-p}}\bullet\frac{2|\varphi_l(z)|}{(1-|\varphi_l(z)|^2)^2}$$

$$=\frac{2^n(2+\alpha)^n}{p^n}\sup_{z\in D^n}\sum_{k=1}^{n}(1-|z_k|^2)^q\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\prod_{l=1}^{n}\frac{1}{(1-|\varphi_l(z)|^2)^{\frac{2+\alpha+p}{p}}}\bullet|\varphi_l(z)|$$

$$=\frac{2^n(2+\alpha)^n}{p^n}|\varphi_l(z)|\sup_{z\in D^n}\sum_{k,l=1}^{n}\frac{(1-|z_k|^2)^q}{\prod_{l=1}^{n}(1-|\varphi_l(z)|^2)^{\frac{2+\alpha+p}{p}}}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|$$

$$\ge\frac{2^n(2+\alpha)^n}{p^n}|\varphi_l(z)|\varepsilon_0 \qquad (3.8)$$

This contradicts with (3.7) and shows that (3.2) holds. The proof is completed.

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