Available online www.jocpr.com

Journal of Chemical and Pharmaceutical Research, 2014, 6(3):47-55

Research Article

ISSN : 0975-7384
CODEN(USA) : JCPRC5

Bezier surfaces on circle and fan-shaped domain

1*Chunfeng Liu, 1Xiaolong Zhang, 1,2Aimin Yang and 1Jincai Chang

1College of Science, Hebei United University, Tangshan, P. R. China
2College of Mechanical Engineering, Yanshan University, Qinhuangdao, China

---

ABSTRACT

The Bernstein basis in Cartesian coordinate system rotated around z can be the rotating Bernstein basis in cylindrical coordinate system, so it can get rotating Bezier surface in cylindrical coordinate system. As this surface has good symmetry, the shape of the rules is not common in geometric modeling, in order to deal this problem we introduce new parameters θ constitute tensor product type Bernstein basis in cylindrical coordinate system, and gives the tensor Bezier surface in unit circle domain which has it nature and de Casteljau algorithm. But there will be a gap in Bezier surfaces, so we popularized Bezier method on circular area to any type on the tensor Bezier surface on fan-shaped area.

Keywords: Cylindrical coordinate system, Bezier surfaces, Rotate, Unit circle domain, Fan-shaped area.

INTRODUCTION

In the 1960s, with the rapid development of automobile industry, the automotive shape aided design aspects urgent to find a practical modeling tool. In 1972, the Bezier in France's Renault car company proposed a curve which is defined by a control polygon method [1-2], only by move the control points you can easily modify the shape of the curve and the shape changes completely expected. Bezier method is simple and solving overall shape of the control problem excellent, so Bezier method plays an important role in CAGD and it took a big step forward for the mathematical description of the shape of industrial products. Later, after Forrest [3] study it found that the Bezier curves are valued Bernstein polynomial form.

American Ryan Aircraft Corporation and the University of Cambridge are applied over Bezier method, the former using Bezier patches (BBP) to establish a system of curves and surfaces in 1972 and the other developed DUCT system. Farin is studied the rational Bezier curve [4]. In China Bezier curves has a lot research, such as Buqing Su [5,6], Dingyuan Liu [5-7], Guozhao Wang[8,9], Fazhong Shi[10] and so on.

This paper is organized as follows: The second part of this article describes some definitions of Bernstein basis and Bezier curve; the third part of this article gives rotating Bernstein basis Bezier surface definition cylindrical coordinates which has it nature and the de Casteljau algorithm. The fourth part of this paper gives the tensor product Bernstein basis and Bezier surface definition on cylindrical coordinate system which has it nature and de Casteljau algorithm. The fifth part of this article general tensor product Bezier surfaces on the unit circle domain to Bezier surface on fan-shaped area.

2. Bezier curves

2.1. Bernstein basis

Given a function $f(t)$ on $[0,1]$, we defined the Bernstein polynomial [14]

$$B_n(f;t)=\sum_{i=0}^{n} f(\frac{i}{n})B_i^n(t), 0 \le t \le 1$$

It called the n-th Bernstein polynomial. Among them

$$B_i^n(t)=\binom{n}{i}t^i(1-t)^{n-i} \qquad i=0,1,\cdots,n; 0\le t\le 1$$

Where the binomial coefficients are given by

$$\binom{n}{i}=\begin{cases}\frac{n!}{i!(n-i)!} & 0\le i\le n\\ 0 & \text{else}\end{cases} \tag{1}$$

$\{B_i^n(t)\}_{i=0}^n$ are known as the Bernstein basis, which are linearly independent and constitute. And each function $B_i^n(t)(i=0,1,\cdots,n)$ is called the Bernstein basis.

2.2. Bezier curves

Give n +1 vector space $P_i \in R^3(i=0,1,2,\cdots,n),$ we get n parametric curve

$$P(t)=\sum_{i=0}^{n} P_i B_i^n(t), 0\le t\le 1$$

Bézier curve degree is n. P_i called control points. Connected with a straight segment between two adjacent control nodes we can get the n-side line polygon called the Bézier polygon or control polygon. Especially when the Bernstein coefficient is the real number b_i, it can desire the control vertices as $P_i=(\frac{i}{n},b_i)(i=0,1,\cdots,n)$.

3. The rotating Bezier surface

In the geometry the surface can be obtained by rotating by curve, such surface tends to have better symmetry. Rotated Bezier curve we can be obtained rotating Bezier surface, Then we can introduced the method to represent freedom curve in geometric modeling to study the rotating Bezier surface on cylindrical coordinate system.

3.1. Rotating Bernstein basis

If $f(r)$ is a function on the unit circle domain, transferred Bernstein basis of Cartesian coordinates around the axis of z we can get rotating Bernstein basis

$$B_i^n(x,y)=\binom{n}{i}\sqrt{x^2+y^2}^{\,i}(1-\sqrt{x^2+y^2})^{n-i} \tag{2}$$

Conversion (2) to cylindrical coordinate system can be obtained

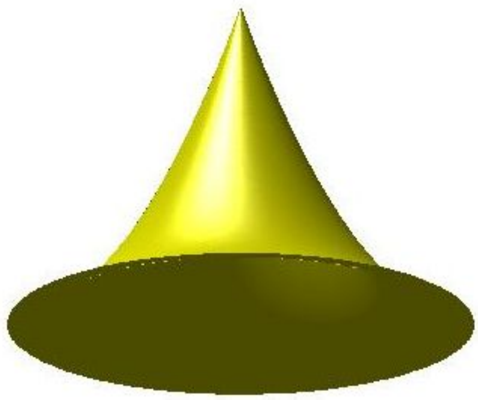
$$B_i^n(r)=\binom{n}{i}r^i(1-r)^{n-i}$$

Called the n-th rotation Bernstein basis

$$B_i^n(r)=\binom{n}{i}r^i(1-r)^{n-i} \qquad i=0,1,\cdots,n; 0\le r\le 1$$

Among them $r=\sqrt{x^2+y^2}$ and combination coefficients $\begin{pmatrix} n \\ i \end{pmatrix}$ still defined as (1)

$\{B_i^n(r)\}_{i=0}^n$ are linearly independent, and Each function $B_i^n(r)(i=0,1,\cdots,n)$ called the rotating Bernstein basis. The follow Fig are rotation Bernstein basis



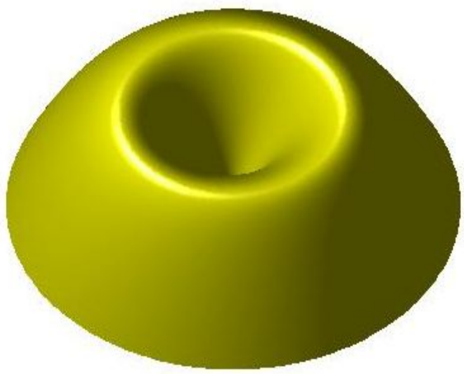


Fig 1 Rotating Bernstein basis $B_2^2(r)$ **Fig 2 Rotating Bernstein basis** $B_1^2(r)$

3.2. The rotating Bernstein basis character

Character 1 non-negative. $B_i^n(r)\geq 0$, for all i,n and $0\leq r\leq 1$;

Character 2 Recurrence relation. $B_i^n(r)=(1-r)B_i^{n-1}(r)+rB_{i-1}^{n-1}(r)$ and $B_0^0(r)\equiv 1$, $B_j^n(r)\equiv 0, j\notin\{0,\cdots,n\}$;

Character 3 Unit decomposition. $\sum_{i=0}^{n} B_i^n(r)=1$;

Character 4 endpoint. $B_0^n(0)=B_n^n(1)=1$;

Character 5 The max. $B_i^n(r)$ will achieved its maximum at $u=i/n$;

Character 6 Derivative. $B_i'^n(r)=\dfrac{dB_i^n(r)}{dr}=n(B_{i-1}^{n-1}(r)-B_i^{n-1}(r))$,among them $B_{-1}^{n-1}(r)\equiv B_n^{n-1}(r)\equiv 0$.

3.3. Bezier surfaces

Given the n +1 vectors $P_i=(r_i,z_i)(i=0,1,2,\cdots,n)$ on cylindrical coordinate system, we call n times parametric surfaces as

$$P(r)=\sum_{i=0}^{n} P_i B_i^n(r), 0\leq t\leq 1 \qquad (3)$$

It is an n-th Bézier surface. P_i called the control curve, the control line is on space of the circle. Connecting adjacent two control curves by turn we can get Bézier control surfaces.

3.4. Rotating Bezier surface charactor

Character 1 Geometric invariance and affine invariance. As Bernstein basis satisfy the unit decomposition we make Bézier surface (3) affine transformation, which uses linear transformation M and translational c role to get new surface

$$P^*(r) = MP(r) + c = M\sum_{i=0}^{n} P_i B_i^n(r) + c\sum_{i=0}^{n} B_i^n(r)$$
$$= \sum_{i=0}^{n} MP_i B_i^n(r) + \sum_{i=0}^{n} cB_i^n(r) = \sum_{i=0}^{n} (MP_i + c)B_i^n(r)$$
$$= \sum_{i=0}^{n} P_i^* B_i^n(r)$$

Affine transformation, the original Bezier surface (3) for the control curve P_i obtained to the new control curve $P_i^* (i = 0,1,2,\cdots,n)$.

Character 2 Convex hull property. The Bernstein basis nature shows that $B_i^n(r)$ constitutes the weight function. For fixed r, $P(r)$ is a weighted average of the control points P_i. From the view of geometric, that means Bezier surface falls in the convex hull of the control surfaces.

Character 3, Derivative function. By the nature of Bernstein basis derivative we can obtained

$$P'(r) = n\sum_{i=0}^{n-1} (P_{i+1} - p_i)B_i^{n-1}(r)$$

The first derivative of n times Bezier surface is an n-1 Bezier surface.

Character 4 Endpoint.

$$P(0) = P_0, P(1) = P_n,$$
$$P'(0) = n(P_1 - P_0), P'(1) = n(P_n - P_{n-1}).$$

Character 5 Symmetry. If you reverse the order of Bezier surface (2) control points, put $P_i^* = P_{n-i} (i = 0,1,\cdots,n)$, then the new Bezier surface is

$$P^*(r) = \sum_{i=0}^{n} P_i^* B_i^n(r) = \sum_{i=0}^{n} P_{n-i} B_i^n(r)$$
$$= \sum_{i=0}^{n} P_i B_{n-i}^n(r) = \sum_{i=0}^{n} P_i B_i^n(1-r) = P(1-r)$$

$P^*(r)$ and $P(r)$ describe the same curve, just through a parameter change $s = 1 - r$, so the curve direction opposite to the original one.

3.5. De Casteljau algorithm of rotating Bezier surface

By the recursive nature of Bernstein basis functions

$B_i^n(r) = (1-r)B_i^{n-1}(r) + rB_{i-1}^{n-1}(r)$, $B_{-1}^{n-1}(r) = B_n^{n-1}(r) \equiv 0, i = 0,1,\cdots,n$

We can get

$$P(r)=\sum_{i=0}^{n}P_iB_i^n(r)$$
$$=P_0(1-r)B_0^{n-1}(r)+\sum_{i=1}^{n-1}P_i[(1-r)B_i^{n-1}(r)+rB_{i-1}^{n-1}(r)]+P_nrB_{n-1}^{n-1}(r)$$
$$=(1-r)\sum_{i=0}^{n-1}P_iB_i^{n-1}(r)+t\sum_{i=0}^{n-1}P_{i+1}B_i^{n-1}(r)$$
$$=\sum_{i=0}^{n-1}((1-r)P_i+rP_{i+1})B_i^{n-1}(r)$$

It can get a n rotating Bezier surface is composed of two $n-1$ Bezier surface. So we can evaluated any point on a Bezier surface by use recursive of de Casteljau algorithm

$$P_i^k(r)=\sum_{j=0}^{k}P_{i+j}B_j^k(r), i=0,1,\cdots n-k$$
$$P(r)=\sum_{i=0}^{n-1}P_i^1B_i^{n-1}(r)=\cdots=\sum_{i=0}^{n-k}P_i^kB_i^{n-k}(r)=\cdots=P_0^n(r)$$

$P_0^n(r)$ Is requested as $P(r)$.

4. The tensor-based Bezier surfaces on circular area

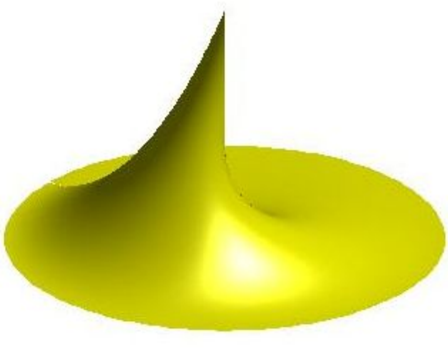
Rotated the Bezier curve on a Cartesian coordinate system obtained rotating Bezier surface which with good symmetry, but in geometric modeling that Bezier surface are rare. Seen from the upper part we found the symmetry of rotating Bezier is result of unknown parameters r , in order to change this symmetry we introduces another new parameter θ , so we can get tensor product Bezier surface model similar Bezier surfaces in Cartesian. The Bezier surface control turn the control line into the control point, which has more closely to the Bezier curve on the Cartesian coordinates.

4.1. Tensor Bernstein basis on Circle domain

In the cylindrical coordinate system, the entire coordinate plane shown as (r,θ,z) , as the space can be expressed by a binary function with parameter r,θ , so we can show this Bernstein basis as the product of two Bernstein basis like that

$$B_{i,j}^{m,n}(r,\theta)=B_i^m(r)B_j^n(\frac{\theta}{2\pi}), i=0,1,\cdots,m, j=0,1,\cdots n$$

The follow Fig give the Bernstein basis tensor images on the unit circle



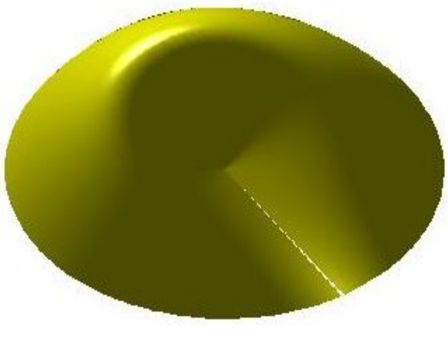


Fig 3 The tensor Bernstein basis of $B_{2,2}^{2,2}(r,\theta)$ **Fig 4 The tensor Bernstein basis of** $B_{1,1}^{2,2}(r,\theta)$

4.2. The character of tensor Bernstein basis

Character 1, Non-negative. $B_{i,j}^{m,n}(r,\theta) \geq 0$, the value on the unit circle.

Character 2, Unit decomposition. $\sum_{i=0}^{m}\sum_{j=0}^{n}(B_{i,j}^{m,n}(r,\theta)) \equiv 1$

Character 3, Derivative

$$\frac{\partial^{p+q}}{\partial r^p \partial \theta^q} B_{i,j}^{m,n}(r,\theta) = (B_i^m(r))^{(p)}(B_j^n(\theta))^{(q)}$$

$$\frac{\partial}{\partial r} B_{i,j}^{m,n}(r,\theta) = (B_i^m(r))'(B_j^n(\theta)) = m(B_{i-1}^{m-1}(r) - B_i^{m-1}(r))B_j^n(\theta)$$

$$\frac{\partial}{\partial \theta} B_{i,j}^{m,n}(r,\theta) = (B_i^m(r))(B_j^n(\theta))' = \frac{n}{2\pi} B_i^m(r)(B_{j-1}^{n-1}(\theta) - B_j^{n-1}(\theta))$$

Character 4, Integral equivalence

$$\int_0^1\int_0^1 B_{i,j}^{m,n}(r,\theta)drd\theta = \frac{1}{(m+1)(n+1)}, i = 0,1,\cdots m, j = 0,1,\cdots,n$$

4.3. Tensor Bezier surfaces on Circle domain

Given $(m+1)(n+1)$ vector $P_{i,j} = (r_i, \theta_j, z_{i,j})$ ($i = 0,1,\cdots m, j = 0,1,\cdots,n$)on cylindrical coordinate system, $B_i^m(r)$, $B_j^n(\theta)$ are Bernstein basis. The tensor Bezier surface defined on the unit circle of cylindrical coordinate system is

$$P(r,\theta) = \sum_{i=0}^{m}\sum_{j=0}^{n} P_{i,j} B_i^m(r) B_j^n(\theta)$$

It called $m \times n$ Bezier surface, $P_{i,j}$ named control vertices, connected to the same type of control vertices In turn can be get the polyline which called control network.

4.4. The character of tensor Bezier surface on Circle domain

As a new Bezier surfaces, it also has similar properties to ordinary Bezier surfaces

Character 1, Geometric invariance and affine invariance.

Character 2, Convex hull property. As the Bernstein basis has the nature of non-negative and units decomposition, then we known tensor Bezier surfaces on cylindrical coordinate system is located inside the convex hull of control mesh.

Character 3, Isoparms character. The isoparametric lines $\theta = \theta^*$ and $r = r^*$ of tensor product Bezier surface are also Bezier curve

$$P(r,\theta^*) = \sum_{i=0}^{m}(\sum_{j=0}^{n} P_{i,j} B_j^n(\theta^*)) B_i^m(r), r \in [0,1]$$

$$P(r^*,\theta) = \sum_{j=0}^{n}(\sum_{i=0}^{m} P_{i,j} B_i^m(r^*)) B_j^n(\theta), \theta \in [0,2\pi]$$

Character 4, Corner interpolation character. The four corners of surfaces interpolations nets of four corner points,

$$P(0,0) = P_{0,0},\ P(0,1) = P_{0,n},\ P(1,0) = P_{m,0},\ P(1,1) = P_{m,n}$$

4.5. The de Casteljau algorithm of tensor Bezier surface

The tensor Bezier surface on cylindrical coordinate system has many similarities with common Bezier surface

$$P(r,\theta)=\sum_{i=0}^{m}\sum_{j=0}^{n}P_{i,j}B_i^m(r)B_j^n(\theta)$$
$$=\sum_{i=0}^{m}\sum_{j=0}^{n}P_{i,j}((1-r)B_i^{m-1}(r)+rB_{i-1}^{m-1}(r))((1-\theta)B_j^{n-1}(\theta)+\theta B_{j-1}^{n-1}(\theta))$$
$$=\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}((1-r)(1-\theta)P_{i,j}+(1-r)\theta P_{i,j+1}+r(1-\theta)P_{i+1,j}+r\theta P_{i+1,j+1})B_i^{m-1}(r)B_j^{n-1}(\theta)$$

This shows a $m\times n$ Bezier surfaces is combination by four $(m-1)\times(n-1)$ Bezier surface. So we can obtained at any point on Bezier surface by de Casteljau algorithm

$$P_{i,j}^{0,0}(r,\theta)\equiv P_{i,j}^{0,0}=P_{i,j}, i=0,1,\cdots,m, j=0,1,\cdots,n$$

$$P_{i,j}^{k,s}(r,\theta)=(1-r)(1-\theta)P_{i,j}^{k-1,s-1}(r,\theta)+(1-r)\theta P_{i,j+1}^{k-1,s-1}(r,\theta)+r(1-\theta)P_{i+1,j}^{k-1,s-1}(r,\theta)+r\theta P_{i+1,j+1}^{k-1,s-1}(r,\theta)$$ a mong them $k=0,1,\cdots,m$, $s=0,1,\cdots,n$, $i=0,1,\cdots,m-k$, $j=0,1,\cdots,n-s$ 。

It can be written as

$$P(r,\theta)=\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}P_{i,j}^{1,1}B_i^{m-1}(r)B_j^{n-1}(\theta)$$
$$=\cdots$$
$$=\sum_{i=0}^{m-k}\sum_{j=0}^{n-s}P_{i,j}^{k,s}B_i^{m-k}(r)B_j^{n-s}(\theta)$$
$$=\cdots$$
$$=P_{0,0}^{m,n}(r,\theta)$$

$P_{0,0}^{m,n}(r,\theta)$ is required point on Bezier surface.

5. The tensor Bezier surfaces on fan-shaped domain

5.1 Bernstein basis on fan-shaped domain

The discussed above tensor Bezier surface model on circular domain is a special surface which has a good closure and discontinuous at the origin, they all was not successful when modeling application so we must have a better general methods. Here we give the tensor Bezier surface modeling on fan-shaped domain, the following figure give the fan-shaped domain, among them o is origin and or is Polar radius, and the blue area is the shape regional.

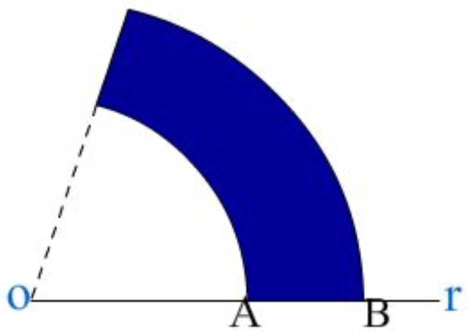


Fig 5 A schematic figure of fan-shaped domain

Among the Fig 5, $OA=r_1$, $OB=r$ and $r-r_1\in[0,1]$, the angle $\theta\in[0,\omega]$. For tensor Bezier surfaces on this area we can be defined two groups Bernstein basis of r,θ is

$$B_i^m(r) = B_i^m(r - r_1), i = 0,1,\cdots,m$$

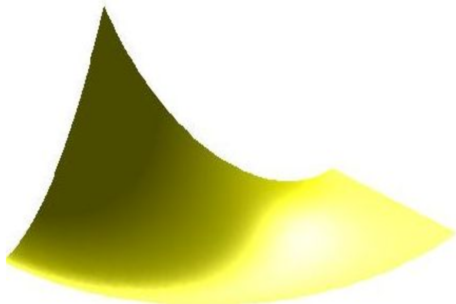
$$B_j^n(\theta) = B_j^n(\frac{\theta}{\omega}), j = 0,1,\cdots n$$

So we can be obtained tensor Bernstein basis through the fan-shaped domain

$$B_{i,j}^{m,n}(r,\theta) = B_i^m(r - r_1)B_j^n(\frac{\theta}{\omega}), i = 0,1,\cdots,m, j = 0,1,\cdots n$$

When $r_1 = 0$, $\omega = 2\pi$, fan-shaped domain is the unit circle domain and when $r_1 \neq 0$, $\omega = 2\pi$ Fan-shaped domain degraded as the unit ring.

When $r_1 = 1$, $\omega = \frac{\pi}{2}$, we can be obtained tensor Bernstein basis on fan-shaped domain as following figure



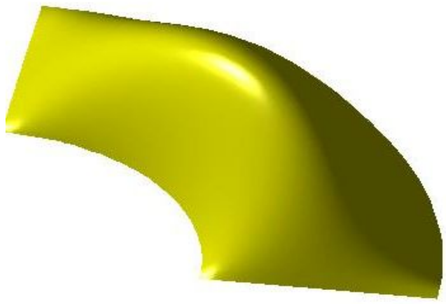


Fig 6 The tensor Bernstein basis of $B_{2,2}^{2,2}(r,\theta)$ **Fig 7 The tensor Bernstein basis of** $B_{1,1}^{2,2}(r,\theta)$

Since the Bernstein basis of unit circle is a special case of the Bernstein basis on fan-shaped domain, so we can easily get the similar Bernstein basis properties on fan-shaped domain.

5.2 Bezier surface on fan-shaped domain

Since the above has been given Bernstein basis on the fan-shaped domain, When we along concentric circle and rays given control mesh $P_{i,j} = (r_i, \theta_j, z_{i,j})$ $i = 0,1,\cdots m, j = 0,1,\cdots,n$ then we can get the fan-shaped domain Bezier surface

$$P(r,\theta) = \sum_{i=0}^{m}\sum_{j=0}^{n} P_{i,j}B_i^m(r)B_j^n(\theta)$$

It can be seen the same manifestations of Bezier surface on the fan-shaped domain and on the unit circle, due to the definition of Bernstein basis differences the resulting final expression is quite different., but they have similar properties and de Casteljau algorithm.

CONCLUSION

In this paper Bernstein basis of Cartesian coordinates rotate along the axis of z to get rotating Bernstein basis, and then obtain rotating Bezier surface of cylindrical coordinate system. However, the rotating Bezier surface with good symmetry in geometry modeling so it not widely used. In order to solve this problem, this paper introduces a new parameter of the angle on cylindrical coordinates to get tensor Bernstein basis on circle domain, so it resulting tensor Bezier surface on circular domain. But this surface would still cause discontinuities, to avoid this problem we introduces tensor Bezier surface on fan-shaped domain instead of on circle domain.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (No.61170317) and the National Natural Science Foundation of Hebei Province (No. E2013209215).

REFERENCES

[1]Bezier,P. The Mathematicas and Applications(Translated by A.R.Forrest), John Wiley and Sons,Londan,**1972**.
[2]Bezier,P .The Mathematics Basis of the UNISURF CAD System, Butterworths, England, **1986**.
[3]Forreste,A.R. *Computer*,Vol 15,No 1.**1972**, 71-79.
[4] Farin.G. *CAD*,Vol.15,No 2,**1983**, 73-77.
[5] Buqing Su, Dingyuan Liu. *Computational Mathematics*, **1980**,2: 289-298.
[6]Buqing Su, Dingyuan Liu. *Applied Mathematics*,**1985**,8(1)：70-83.
[8]Guozhao wang, Jinfu Shen. *Journal of Zhejiang University*,**1985**,19(3):123-130.
[9]Guojin Wang,Guozhao Wang,Jianmin Zheng. Computer aided geometric design. Beijing: Higher education press, **2001**.
[10]Fazhong Shi,Junheng Wu. *Journal of aviation*,**1982**, 197-104.
[11]G. Celniker, D. Gossard. *Computer Graphics*,**1991**, 25(4)：257-266.
[12]Young Joon Ahn, Christoph Hoffmann. *Journal of Mathematical Analysis and Applications*, Volume 410, Issue 1, 1 February **2014**, 257-266.
[13]Busé Laurent. *Computer-Aided Design*, Volume 46, January **2014**, 14-24.
[14]Orest Shardt, John C. Bowman. *Computer-Aided Design*, Volume 44, Issue 5, May **2012**, 484.e1-484.e10.
[15]Stanisław Lewanowicz, Paweł Woźny. *Applied Mathematics and Computation*, Volume 217, Issue 9, 1 January **2011**, 4596-4611.
[16]Paweł Woźny, Stanisław Lewanowicz. *Journal of Computational and Applied Mathematics,* Volume 235, Issue 3, 1 December **2010**, 785-804.
[17]R.T. Farouki, N. Szafran, L. Biard. *Computer-Aided Design*, Volume 41, Issue 11, November **2009**, 772-781.
[18]Xi-An Han, YiChen Ma, XiLi Huang. *Journal of Computational and Applied Mathematics*, Volume 217, Issue 1, 15 July **2008**, 180-193.