Available online www.jocpr.com

Journal of Chemical and Pharmaceutical Research, 2016, 8(8):650-668

Research Article

ISSN : 0975-7384
CODEN(USA) : JCPRC5

Analytical investigation of Hatzikiriakos and power law slip effects on forced convection of Phan-Thien-Tanner fluids between parallel plates at high zeta-potentials

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ABSTRACT

The present study deals with the effect of slip on the heat transfer and entropy generation characteristics of viscoelastic fluids in a channel. The slip has been modeled using three different slip laws namely, Navier's non-linear slip law, Hatzikiriakos slip law and asymptotic slip law. The viscoelastic nature of the fluid is captured using the linear version of simplified Phan-Thien-Tanner (s-PTT) model. The flow is assumed to be hydrodynamically and thermally fully developed with uniform heat flux boundary condition at the wall. Viscous dissipation is included while axial conduction is ignored. The governing equations have been solved analytically and the reasons behind the observed trends have been explained in detail. Specifically, Nusselt number shows a complex dependence on the viscoelastic group, slip coefficients and the pressure gradient. Finally, a comparison between Hatzikiriakos slip law and the asymptotic slip law shows that the slip velocity and consequently the Nusselt number is higher for Hatzikiriakos version of slip law

Keywords: Phan-Thien-Tanner; entropy generation; convective heat transfer; slip laws; non-linear Navier; Hatzikiriakos

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INTRODUCTION

There are numerous processes in the industry where the fluid flow depicts a viscoelastic nature. A prominent example is the extrusion process in the polymer industry. Here the gap between the barrel and the screw of the extruder is small and thus the concomitant flow can be modeled as the flow between two parallel plates [1]. Moreover, the flow of melts in pipes (before extrusion) takes place at elevated temperatures which warrants an investigation into their heat transfer characteristics.

As mentioned, the rheological behavior of the fluid in such processes exhibits viscoelastic characteristics. One of the most common mathematical models to simulate viscoelastic behavior is the (simplified) Phan-Thien-Tanner (s-PTT) model [2-3] which is the focal point of study in this paper. Additionally, the slip of the fluid at solid boundaries is a very common phenomena in polymer processing, affecting the quality of the final product [4].

There have been numerous studies to investigate the flow characteristics of viscoelastic fluids. These include studies related to Couette flow [5-7], channel and pipe flow [8], Couette –Poisuеille flow [9] and annular flow [10]. On the other hand, there has been only one study which takes into account the effect of slip on the hydrodynamics of viscoelastic fluids- a very extensive and detailed study by Ferras et. al [11].

Compared to the hydrodynamics, the heat transfer of flow of viscoelastic fluids has not been studied extensively in literature. There have been some studies pertaining to heat transfer of viscoelastic fluids in channels subject to uniform heat flux [12], uniform wall temperature [13] and with moving upper plate [1]. In all these papers, viscous dissipation has been accounted for as it is very relevant in polymer melts. There has also been a study of the Graetz problem with viscoelastic fluids [14]. But there has been no investigation in extant literature of the effect of slip on the heat transfer characteristics of viscoelastic fluids.

Investigation of entropy generation in a system is important because entropy generation determines the amount of “lost work” in a system [15]. To the best knowledge of the author, there has been absolutely no investigation of entropy generation in a viscoelastic fluid.

The aim of this paper is to study the heat transfer and entropy generation characteristics of the viscoelastic fluids modeled with s-PTT model and subject to different types of slip at the wall. The flow is hydrodynamically and thermally fully developed, viscous dissipative and subject to uniform heat flux boundary condition. This paper is the first study of the effect of slip on the heat transfer characteristics of viscoelastic fluids. Moreover, it is the first study of the entropy generation characteristics of a viscoelastic fluid, with or without slip. In this respect, the paper fills a void in archival literature.

EXPERIMENTAL SECTION

The remainder of this paper is arranged as follows. In Section 2, the slip laws used in this study have been introduced and described in detail. In Section 3, the relevant conservation equations of mass, momentum and energy have been solved taking into account the slip laws. Expressions for velocity distribution, shear stress distribution, temperature distribution, Nusselt number, entropy generation distribution and Bejan number have been presented in this section. Section 4 gives a detailed explanation of the results and the reasons for the trends observed. Section 5 is the conclusion of this study. The no slip boundary condition is the most general velocity boundary condition used in literature to model the velocity at solid-fluid interface. But despite its ease of use (and the consequent ubiquity), this boundary condition is an assumption and cannot be derived from first principles [16].

A fluid is said to slip when there is a non-zero tangential component of fluid velocity relative to the surface in contact with it. The nature of slip depends on many factors like surface roughness, chemical composition of fluids, presence of dissolved gases, characteristic length of the flow etc.

A slip law relates the slip velocity of the fluid to the physical characteristics of the flow. The earliest slip law was proposed by Navier [17], and later rigorously derived for gases by Maxwell [18]. The slip law introduced a new parameter called slip length/slip coefficient which correlates the slip velocity with the velocity gradient/ shear stress at the wall. This slip law is generally used for Newtonian fluids and its use for non- Newtonian fluids has been scarcely reported in literature.

An advanced version of Navier’s slip law is Navier’s non-linear slip law which is given by:

$$u_w = (\mp\tau_{yx,w})^m k_{nl} \tag{1}$$

For bottom wall, “+” sign is used, since shear stress is positive there (according to the position of the coordinate axes shown in Fig. 1 and explained in next section). Similarly, $"-"$ sign is used for top wall. k_{nl} is the non- linear slip coefficient while *m* (>0)is the slip law exponent. For *m*=1, the linear Navier’s slip law is recovered. This slip law has been able to correctly model the slip for many experimental conditions of Poisseuille flow [19-21].

Navier’s non- linear slip law predicts that slip will start as soon as the fluid flows, i.e for any non zero value of shear stress. Hatzikiriakos proposed a slip law which delays the onset of slip until the shear stress at the walls exceeds a critical stress [22] . The slip law proposed by him is derived from Eyring’s theory of liquid viscosity. It is aptly named as Hatzikiriakos slip law and is formulated for one-dimensional flow as:

$$u_w = \begin{cases} k_{H1}\sinh(\mp k_{H2}\tau_{yx,w} - \tau_c) & ,\left|\tau_{yx,w}\right| > \tau_c \\ 0 & ,\left|\tau_{yx,w}\right| < \tau_c \end{cases} \tag{2}$$

In literature, there are various instances where Hatzikiriakos slip law was used to model the slip in channel flows [11, 22].

The third slip law employed in this research paper is the asymptotic slip law [11, 23] given by:

$$u_w = k_{A1}\ln(\mp k_{A2}\tau_{yx,w} + 1) \tag{3}$$

In these equations, the slip coefficients $k_{nl}, m, k_{H1}, k_{H2}, k_{A1}, k_{A2}$ are experimentally determined for different fluids. They control the amount of slip at the wall and influence the shape of the slip velocity vs shear stress curve. The governing equations of the viscoelastic flows consist in the continuity equation ($\rho\nabla.u = 0$ conservation of mass) $\nabla \cdot \boldsymbol{u} = 0,$ and the momentum equation ($\rho(u.\nabla)u = \nabla.(-pI + \mu[\nabla u + (\nabla u)^T] + F)$ conservation of linear momentum) $\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) = -\nabla p + \mu_S \nabla^2 \boldsymbol{u} + \nabla \cdot \boldsymbol{\sigma},$ where u is the velocity vector, p is the isostatic pressure and σ the viscoelastic extra-stress tensor. The left hand side in the equation corresponds to the inertial effects; the three terms on the right hand side of equation are the contributions of the pressure gradient, the Newtonian viscous stress of the solvent, and the viscoelastic stress of the polymers, respectively. Finally, the equations of conservation are supplemented with a constitutive model which closes the system of equations. We use a generic partial-differential viscoelastic model of general form $\frac{\partial \boldsymbol{\sigma}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\sigma} - \left(\boldsymbol{\sigma} \cdot \nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}} \cdot \boldsymbol{\sigma}\right) + \frac{f(\boldsymbol{\sigma})}{\lambda}\boldsymbol{\sigma} = \frac{2\mu_P}{\lambda}\dot{\boldsymbol{\varepsilon}},$ where f (σ) is a relaxation function, and $\dot{\boldsymbol{\varepsilon}} = (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})/2$ is the strain rate tensor. Depending on the expression of f (σ), popular viscoelastic models can be recovered [1], see table 1. The material parameters ρ, μS, μP and λ are the density, the solvent viscosity, the polymer viscosity and the relaxation time, respectively. Table 1 defines the terms mentioned above. ε is Elasticity of the fluid, parameter in PTT model; ΔP/L Predicted drop in pressure for fully developed flow per unit length using the Phan-Thien-Tanner (PTT) model; $\Delta Pexcess_1$isThe difference of the measured drop in pressure through the model and thedrop in pressure that occurs when We = 0; ΔPexcess2 is the difference of the measured drop in pressure through the model and the fully developed pressure drop in PTT; τxx Shear stress at the inlet, also the normal stress at the inlet. We: is Weissenberg Number: λ<u>/y; Where λ is the time constant that describes how fast the polymer "forgets" its shape, <u> is the average velocity, and y is half the height of the model.

Table 1. Expressions of the relaxation function *f (σ)* in the generic constitutive equation (4), for different viscoelastic models

Viscoelastic model	Relaxation function $f(\boldsymbol{\sigma})$
Oldroyd-B	1
Giesekus	$1+(\alpha\lambda/\mu_1)\boldsymbol{\sigma}$
Linear PTT	$1+(\varepsilon\lambda/\mu_1)\mathrm{tr}(\boldsymbol{\sigma})$
Exponential PTT	$\exp[(\varepsilon\lambda/\mu_1)\mathrm{tr}(\boldsymbol{\sigma})]$
FENE-CR	$[1+(\lambda/\mu_1 L^2)\mathrm{tr}(\boldsymbol{\sigma})]^{-1}$

In dimensionless forms:

$$\sigma_x^* = \frac{\sigma_x}{n/2\lambda} = \frac{1-\sqrt{1-\xi(2-\xi)\tau^{*2}}}{\xi}$$

and

$$\tau^* = \frac{p_x y}{n/2\lambda}$$

$$\lim \sigma_x^* = \tau^{*2}$$

where $\xi \to 0$

So $f = 1 + \varepsilon\frac{\sigma_x^* + \sigma_y^*}{2}$ in Linear PTT and $\sigma_y = \sigma_x \frac{-\xi}{2-\xi}$. In a same way :

$$\rightarrow f = 1 + \frac{1-\xi}{2-\xi}\varepsilon\sigma_x^*$$

Which has a solution of

$$u_y\Big|_{\xi=0} = \gamma = \frac{1}{\lambda(2-\xi)}\left(\tau^* + \frac{1-\xi}{2-\xi}\varepsilon\tau^{*3}\right)$$

or

$$u = \frac{p_x}{\eta}y^2 + \frac{\varepsilon\lambda^2 p_x^3}{\eta^3}y^4 + u_{\max}$$

while for this research, we non-dimensionalized every quantity; that with our boundary conditions (described in the next section) give us the following relations:

$$u = \frac{p_x}{\eta}\mathrm{Y}^2 + \frac{\tau_w}{\eta}\mathrm{Y} +$$

$$\frac{\varepsilon\lambda^2}{\eta^3}\left(p_x^3\mathrm{Y}^4 + 4p_x^2\tau_w\mathrm{Y}^3 + 6p_x\tau_w^2\mathrm{Y}^2 + 4\tau_w^3\mathrm{Y}\right)$$ is comparable to $u = \frac{p_x}{2\mu}y(y-h)$

To use the correlations above, the term,ΔP must be calculated at the given ε and We. The first normal stress difference, N, is calculated as N = τxx - τyy, but these stresses are measured at the inlet where τyy = 0, so N = τxx. from continuity $V_y = -u_x \rightarrow V = -\frac{p_{xx}}{2\mu}\left(\frac{y^3}{3} - \frac{y^2 h}{2} + c\right)L$ With BC of $C \rightarrow V(0) = 0$ and from symmetry ($p_x\big|_0 = 0$) at BC $V(h) = -V_m = \frac{p_{xx}h^3}{12\mu} \rightarrow p = \frac{-6\mu V_m}{h^3}x^2 + p_0$ for half length $\frac{p - p_\infty}{\mu V_m / h} = 6\left(\frac{x}{h}\right)^2 \xrightarrow{x=l/2} \Delta p^*_{\max} = \frac{3}{2}\left(\frac{L}{h}\right)^2$ with the velocity of $\frac{u}{V_m} = -6\frac{x}{h}\frac{y}{h}\left(\frac{y}{h} - 1\right)$ and $\frac{V}{V_m} = 2\left(\frac{y}{h}\right)^3 - 3\left(\frac{y}{h}\right)^2$ As $\xi \rightarrow 0$ we have $u = \frac{p_x}{\eta}\left(y^2 - H^2\right) + \frac{\varepsilon p_x^3\lambda^2}{\eta^3}\left(y^4 - H^4\right)$ So we should change $\rho \equiv \mathrm{Re},\ \mu \equiv \mu_s$ to solve $\mathrm{Re}(u.\nabla)u = \nabla.(-pI + \mu_s\left[\nabla u + (\nabla u)^T\right] + T)$ Also the general form PDE for the solution of

$$\frac{\partial}{\partial t}\begin{bmatrix} T_x & T_{xy} \\ T_{xy} & T_y \end{bmatrix} =$$
$$-Wi\left((u.\nabla)T + \left[(\nabla u)T + T(\nabla u)^T\right]\right)$$
$$+\mu_p\left[(\nabla u) + (\nabla u)^T\right] \quad (1)$$

Where is for each component

$$u.\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \quad (2)$$

The extra stress contribution due to the polymer is given by the following Oldroyd-B constitutive relation:

$$T + \lambda\frac{\Delta T}{\Delta t} = 2\eta_p e(u) \quad (3)$$

where the upper convective derivative operator (or Oldroyd derivative) is defined as

$$\frac{\Delta T}{\Delta t} = \frac{\partial T}{\partial t} + (u.\nabla)T - \left[(\nabla u)T + T(\nabla u)^T\right] \tag{4}$$

The polymer is characterized by two physical parameters: The viscosity ηp and the relaxation time λ. The fluid is treated as incompressible with a constant density ρ; the flow equations read: $\rho\frac{\partial u}{\partial t} + \rho(u.\nabla)u = \nabla.\sigma, \nabla.\sigma = 0$ .The extra stress tensor is symmetric: $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{121} & T_{22} \end{bmatrix}$

Therefore, it is necessary to solve three additional equations, for the three components in Equation 3, together with three equations given by Equation 5 for the pressure and two velocity components.

The Weissenberg number is defined as:

$$Wi = \lambda\frac{U_{in}}{R} \tag{5}$$

where Uin is the average fluid velocity at the inlet, R is the radius of the cylinder, and λ is the polymer relaxation time. (An alternative name, which is often used for this nondimensional parameter, is the Deborah number.) A zero Weissenberg number gives a pure viscous fluid (no elasticity) while an infinite Weissenberg number limit corresponds to purely elastic response. Due to the convective nature of the constitutive relation, the solution stability is lost with the increasing fluid elasticity. In practice, already the values Wi > 1 are considered as a high Weissenberg number for many flows of an Oldroyd-B fluid. By adding least squares-type stabilization terms to the Galerkin finite element formulation, you can improve stability and obtain solutions over a larger range of Weissenberg numbers compared to a standard Galerkin formulations. The present model makes use of such least-squares stabilization technique. The flow is stationary, and the problem becomes dimensionless by using R, Uin, and the total viscosity η = ηs + ηp. The nondimensional equations system is the following:

$$\begin{aligned} &\mathrm{Re}(u.\nabla)u = \nabla.(-pI + \mu_s\left[\nabla u + (\nabla u)^T\right] + T) \\ &T + Wi\left((u.\nabla)T - \left[(\nabla u)T + T(\nabla u)^T\right]\right) \\ &= \mu_p\left[(\nabla u) + (\nabla u)^T\right] \end{aligned} \tag{6}$$

where the Reynolds number is Re = R Uinη / ρ, and the relative viscosities of the solvent and polymer are, respectively: μs = ηs / η and $\mu_p = \frac{\eta_p}{\eta} = 1 - \mu_s$. Weak formulation of the above equations and the stabilization terms needed for the extra stress equations are not shown here. Because of the flow symmetry, you model only the upper halves of the channel the cylinder. At the channel centerline, use the symmetry conditions of zero normal flow $u.n = 0$ and zero total tangential stress: $(\sigma.n).t = 0$ where **n** and **t** are the boundary unit normal and tangent vectors, respectively. On a horizontal line of symmetry, the latter condition is reduced to $2\mu_s e_{12} + T_{12} = 0$. At the channel walls and the cylinder surface, the model uses no slip conditions for the velocity together with the condition for the normal component of the extra stress:

$(T.n).n = 0$ .Polymers are unable to exert a normal force on the wall because no polymer can span the wall boundary. *Half-Inlet* Here you specify the developed parabolic velocity profile and the corresponding extra stresses components:

$$u = \frac{3}{2}(1 - s^2)$$

$$T_{11} = 2\mu_p Wi\left(\frac{\partial u}{\partial y}\right)^2$$

$$T_{12} = \mu_p\left(\frac{\partial u}{\partial y}\right)$$

$$T_{22} = 0 \tag{7}$$

where the geometry edge parameter *s* varies from 0 to 1 along the half-inlet boundary. *Outlet* At the outlet, use the pressure boundary condition for developed flow; the only stress acting at the boundary is due to the pressure force p_{out}:

$$\sigma.n = -p_{out} n \tag{8}$$

A schematic of the physical model under study is shown in Fig.1. There are two parallel plates separated by a gap of width *2h*. The fluid flow between the plates is maintained by a constant pressure gradient of p_x, which is numerically a negative quantity. Uniform and equal heat flux *q* is imposed on the two plates. The heat flux is modeled as positive into the walls. The fluid is viscoelastic and its stress-strain relationship is captured by the simplified Phan-Thien-Tanner model. The flow is hydrodynamically and thermally fully developed. The slip at the wall is captured by Navier's non- linear slip law, Hatzikiriakos slip law and asymptotic slip law.

The x- coordinate is along the centre line of the channel while the y-coordinate is perpendicular to the centre line pointing towards the upper plate. The fluid flow occurs in the direction of positive x.

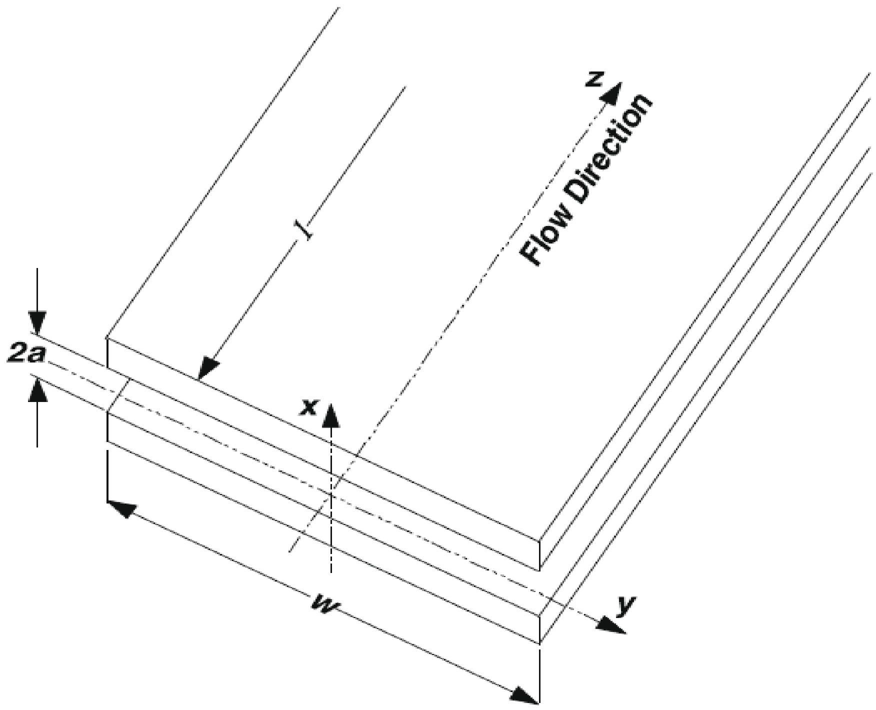

Fig. 1: Schematic of the physical model under study

In this paper, we wish to study phenomena related to viscoelastic fluids. These fluids show characteristics of elastic solids as well as viscous fluids. Whether a fluid would respond elastically or viscously will depend not only on its structure but also on the kinematic conditions it experiences. A viscoelastic material will return to its original shape when the external stress has been removed (elastic response) but it will take some time to do so (viscous response).

Some amount of stored energy will be dissipated (viscous response) while some will be recovered (elastic response). For viscoelastic fluids, shearing motion may give rise to stresses in normal direction too.

The simplified Phan-Thien-Tanner model of visocelastic fluid behavior has been used here. This model is derived from the network theory and has been used extensively in literature to replicate viscoelastic fluid behavior [2-3,8-13]. For this model, the constitutive equation is given by [2-3]:

$$f(tr\boldsymbol{\tau}) + \lambda\breve{\boldsymbol{\tau}} = \eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}) \tag{9}$$

Here, *f* is a function of trace of the stress tensor, λ is the relaxation time, η is the viscosity coefficient and $\breve{\boldsymbol{\tau}}$ stands for upper convective derivative as shown below:

$$\breve{\boldsymbol{\tau}} = \frac{\partial\boldsymbol{\tau}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\tau} - [(\nabla\mathbf{u})^{\mathrm{T}}\cdot\boldsymbol{\tau} + \boldsymbol{\tau}\cdot\nabla\mathbf{u}] \tag{10}$$

The function *f* can take two forms- linear and exponential. The linear form [3] is given by:

$$f(tr\boldsymbol{\tau}) = 1 + \frac{\varepsilon\lambda}{\eta}tr\boldsymbol{\tau} \tag{11}$$

While the exponential form is given by [2]:

$$f(tr\boldsymbol{\tau}) = \exp\left(\frac{\varepsilon\lambda}{\eta}tr\boldsymbol{\tau}\right) \tag{12}$$

Here ε is the elongation parameter which controls the shear thinning behavior of the fluid. Higher is the value of ε, more will be the shear thinning behavior of the fluid. The linear form is a linearized version of the exponential form and is accurate when the term in the brackets of left hand side of Eq. (9) is small. According to Tanner [24], the linear form is accurate when the molecular deformation is small, i.e. in the case of weak flows. Pipe flows and channel flows are weak flows where the results predicted by linear form will be close to the exponential form. For this reason, in this research article only linear form of s-PTT model has been considered and the results presented thereof.

RESULTS AND DISCUSSION

The heat transfer and entropy generation characteristics of the exponential model will be studied in a separate, upcoming paper.

For the EDL layer we have

$$\frac{d^2\psi}{dx^2} = \frac{2n_0 z_v e}{\varepsilon\varepsilon_0}\sinh\left(\frac{z_v e\psi}{k_b T}\right) \tag{13}$$

$$\overline{\psi} = \frac{z_v e\psi}{k_b T} \qquad \overline{x} = \frac{x}{a} \quad \text{and} \quad k = \sqrt{\frac{2n_0 z_v^2 e^2}{\varepsilon\varepsilon_0 k_b T}} \tag{14}$$

$$\frac{d^2\overline{\psi}}{d\overline{x}^2} = (ka)^2\sinh(\overline{\psi}) \tag{15}$$

$$\frac{d\overline{\psi}}{d\overline{x}} = 0 \text{ at } \overline{x} = 0 \quad \text{and } \overline{\psi} = \overline{\xi} \text{ at } \overline{x} = 1 \text{ where } \overline{\xi} = \frac{z_v e\xi}{k_b T} \tag{16}$$

$$\frac{d\overline{\psi}}{d\overline{x}} = \sqrt{2}(ka)\left[\cosh(\overline{\psi}) - \cosh(\overline{\psi}_0)\right]^{1/2} \tag{17}$$

$$\overline{\psi} = 4\tanh^{-1}\left[\tanh\left(\frac{\overline{\xi}}{4}\right)e^{-ka(1-\overline{x})}\right] \tag{18}$$

For the case of planar, fully developed flow considered here, the equation for momentum balance reduces to:

$$\frac{d\tau_{yx}}{dy} = p_x \tag{19}$$

Integrating once, we get:

$$\tau_{yx} = p_x y + c_1 \tag{20}$$

The constitutive equations for the simplified PTT model reduce to:

$$f(\tau_{xx} + \tau_{yy})\tau_{xx} = 2\lambda\tau_{yx}\left(\frac{\partial u}{\partial y}\right) \tag{21}$$

$$f(\tau_{xx} + \tau_{yy})\tau_{yy} = 0 \tag{22}$$

$$f(\tau_{xx} + \tau_{yy})\tau_{xy} = \eta\left(\frac{\partial u}{\partial y}\right) + \lambda\tau_{yy}\left(\frac{\partial u}{\partial y}\right) \tag{23}$$

In Eq. (13) , either $f(\tau_{xx} + \tau_{yy})$ is 0 or τ_{yy} =0. If $f(\tau_{xx} + \tau_{yy})$ =0, unrealistic results are obtained. This leads to τ_{yy} =0.

After substituting τ_{yy} =0, divide Eq. (14) by Eq. (12) to get the following

$$\tau_{xx} = \frac{2\lambda}{\eta}\left(\tau_{yx}\right)^2 \tag{24}$$

So, the set of governing equations reduces to:

$$\tau_{xy} = p_x y + c_1 \tag{25}$$

$$\tau_{xx} = \frac{2\lambda}{\eta}\left(\tau_{yx}\right)^2 \tag{26}$$

$$\tau_{yy} = 0 \tag{27}$$

$$f(\tau_{xx} + \tau_{yy})\tau_{yx} = \eta\left(\frac{\partial u}{\partial y}\right) \tag{28}$$

The above equations are non-dimensionalized with the following dimensionless variables:

$$y' = y/h, \quad x' = \frac{x}{h}, c'_1 = \frac{c_1}{\left(\frac{\eta U}{h}\right)}, \; u' = \frac{u}{U}, G = \frac{p_x}{\left(\frac{\eta U}{h^2}\right)}, \tau'_{yx} = \frac{\tau_{yx}}{\left(\frac{\eta U}{h}\right)} \; \tau'_{xx} = \frac{\tau_{xx}}{\left(\frac{\eta U}{h}\right)}$$

And $De = \dfrac{\lambda U}{h}$ (29)

Here, *De* is the Deborah number; it is the ratio of elastic behavior and the viscous behavior of the flow. U is the characteristic velocity, mostly the average velocity of the flow.

Since the analytical solution would be

$$\mu\frac{d^2 v_z}{dx^2} - \frac{dp}{dz} + E_z\rho = 0 \tag{30}$$

$$\rho = -2(n_0 z_v e)\sinh\left(\frac{z_v e\psi}{k_b T}\right) \tag{31}$$

$$G_1 = \frac{n_0 k_b T}{L\left(-\frac{dp}{dz}\right)} \qquad \bar{v}_z = \frac{v_z}{v_0} \qquad \text{and } \bar{E}_s = \frac{LE_z}{\xi} \tag{32}$$

$$\frac{d^2\bar{v}_z}{d\bar{x}^2} - \frac{2G_1\bar{\xi}\bar{E}_s}{(ka)^2}\frac{d^2\bar{\psi}}{d\bar{x}^2} + 1 = 0 \quad (33)$$

$$\frac{d\bar{\psi}}{d\bar{x}} = \frac{d\bar{v}_z}{d\bar{x}} = 0 \text{ at } \bar{x} = 0 \text{ and } \bar{\psi} = \bar{\xi} \text{ and } \bar{v}_z = 0 \quad \bar{x} = 1 \quad (34)$$

$$\bar{v}_z = \frac{1}{2}(1-\bar{x}^2) - \frac{2G_1\bar{\xi}^2\bar{E}_s}{(ka)^2}\left(1 - \frac{\bar{\psi}}{\bar{\xi}}\right) \quad (35)$$

$$I_s = \int_{A_c} v_z \rho dA_c \quad (36)$$

$$\bar{\rho} = \frac{\rho}{n_0 z_v e} = -2\sinh(\bar{\psi}) \text{ and } \bar{I}_s = \frac{I_s}{2v_0 n_0 z_v ea} \quad (37)$$

$$\bar{I}_s = -2\int_0^1 \bar{v}_z \sinh(\bar{\psi}) d(\bar{x}) \quad (38)$$

$$\bar{I}_s = (I_1 - I_3) - \left(\frac{4G_1\bar{\xi}\bar{E}_s}{(ka)^2}\right)(I_2 - I_3) \quad (39)$$

$$I_1 = \frac{4}{(ka)^3}\left\{-ka\left[\ln\left(\frac{1+\eta e^{ka}}{1-\eta e^{ka}}\right) + \frac{\eta e^{ka}}{(\eta e^{ka})^2 - 1}\right] + \left[Li_2(\eta e^{ka}) - Li_2(-\eta e^{ka})\right] - \left[Li_2(\eta) - Li_2(-\eta)\right]\right\} \quad (40)$$

$$I_2 = \frac{8}{ka}\left[\frac{1 - 2\eta e^{ka}\tanh^{-1}(\eta e^{ka})}{(\eta e^{ka})^2 - 1} - \frac{1 - 2\eta\tanh^{-1}(\eta)}{\eta^2 - 1}\right] \quad (41)$$

$$I_3 = -\frac{4}{ka}\left[\frac{\eta e^{ka}}{(\eta e^{ka})^2 - 1} - \frac{\eta}{\eta^2 - 1}\right] \quad (42)$$

$$\eta = \tanh(\bar{\xi}/4)e^{-ka} \quad (43)$$

$$Li_2(\beta) = -\int_0^{\beta}\frac{\ln(1-t)}{t}dt \quad (44)$$

$$I_c + I_s = 0 \quad (45)$$

$$\bar{I}_c + \frac{G_2(ka)^2}{\bar{\xi}}\bar{I}_s = 0 \quad (46)$$

$$\bar{E}_s = \frac{\frac{G_2(ka)^2}{\bar{\xi}}(I_3 - I_1)}{1 - 4G_1G_2(I_2 - \bar{\xi}I_3)} \quad (47)$$

By using the dimensionless variables, we get the non-dimensional form of the governing equations as shown below:

$$\tau'_{xy} = Gy' + c'_1 \quad (48)$$

$$\tau'_{xx} = 2De\left(Gy' + c'_1\right)^2 \quad (49)$$

$$\frac{\partial u'}{\partial y'} = \left[1 + 2\varepsilon De^2[Gy' + c'_1]^2\right]\left[Gy' + c'_1\right] \quad (50)$$

The velocity boundary conditions are symmetric, i.e. the same boundary conditions are applied on the lower and upper plate. This means $c'_1 = 0$. So,

$$\frac{\partial u'}{\partial y'} = \left[1 + 2\varepsilon De^2[Gy']^2\right]\left[Gy'\right] \quad (51)$$

We integrate the above equation once to get:

$$u'(y') = \frac{Gy'^2}{2} + 2\varepsilon De^2 G^3 \frac{y'^4}{4} + c'_2 \quad (52)$$

Here, c'_2 is an unknown constant whose value will depend on the boundary condition/slip law employed. The non-dimensional form of Navier's non- linear slip law is given by:

$$u'_w = (\mp\tau'_{yx,w})^m k'_{nl} \text{ where, } k'_{nl} = \frac{k_{nl}U^{m-1}\eta^m}{h^m} \quad (53)$$

At y'=1, the above equation reduces to:

$$u'_w = (-G)^m k'_{nl} \quad (54)$$

Using Eq. (25) and Eq. (27) we obtain the following:

$$(-G)^m k'_{nl} = \frac{G}{2} + \frac{2\varepsilon De^2 G^3}{4} + c'_2 \quad (55)$$

$$\Rightarrow c'_2 = (-G)^m k'_{nl} - \frac{G}{2} - \frac{2\varepsilon De^2 G^3}{4} \quad (56)$$

On the other hand, the non-dimensional form of Hatzikiriakos slip law is given by:

$$u'_w = k'_{H1} \sinh\left[\mp(k'_{H2}\,\tau'_{yx,w})\right] \quad (57)$$

Here, $k'_{H1} = \frac{k_{H1}}{U}, k'_{H2} = \frac{k_{H2}\eta U}{h}$ (58)

At upper plate, y'=1, the equation reduces to:

$$u'_w = k'_{H1} \sinh\left[-(k'_{H2}\,G)\right] \quad (59)$$

From Eq. (25) and Eq. (32), we obtain the following value of c'_2 for Hatzikiriakos slip law:

$$c'_2 = k'_{H1} \sinh\left[-(k'_{H2}\,G)\right] - \frac{G}{2} - \frac{2\varepsilon De^2 G^3}{4} \quad (60)$$

Following the same procedure for non-dimensional form of asymptotic slip law:

$$u'_w = k'_{A1} \ln(\mp k'_{A2}\,\tau'_{yx,w} + 1) \quad (61)$$

Here,

$$k'_{A1} = \frac{k_{A1}}{U}, k'_{A2} = \frac{k_{A2}\eta U}{h} \quad (62)$$

which gives the value of c'_2 as

$$c'_2 = k'_{A1} \ln(-k'_{A2}\,G + 1) - \frac{G}{2} - \frac{2\varepsilon De^2 G}{4} \quad (63)$$

Summarizing, expression for non-dimensional velocity profile between the two plates is given by Eq. (25):

$$u'(y') = \frac{Gy'^2}{2} + 2\varepsilon De^2 G^3 \frac{y'^4}{4} + c'_2$$

Where,

$$c'_2 = \begin{cases} (-G)^m k'_{nl} - \frac{G}{2} - \frac{2\varepsilon De^2 G^3}{4} & \text{Navier's non-linear slip law} \\ k'_{H1} \sinh\left[-(k'_{H2}\,G)\right] - \frac{G}{2} - \frac{2\varepsilon De^2 G^3}{4} & \text{Hatzikiriakos slip law} \\ k'_{A1} \ln(-k'_{A2}\,G + 1) - \frac{G}{2} - \frac{2\varepsilon De^2 G}{4} & \text{asymptotic slip law} \end{cases}$$

(64)

The two dimensional energy equation for the thermally and hydrodynamically fully developed flow of a viscous dissipative fluid where axial conduction is neglected is given by:

$$\rho c_p u \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial y^2} + \phi \tag{65}$$

Here, ϕ is the volumetric viscous dissipation rate in the flow. For the flow under consideration, it is given by

$$\phi = \tau_{yx} \frac{\partial u}{\partial y} \tag{66}$$

This reduces the Eq. (27) to

$$\rho c_p u \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial y^2} + \tau_{yx} \frac{\partial u}{\partial y} \tag{67}$$

The pertinent (thermal) boundary conditions for the above equation are:

$$-k \left.\frac{\partial T}{\partial y}\right|_{y=-h} = q \, , k \left.\frac{\partial T}{\partial y}\right|_{y=h} = q \tag{68}$$

Since the flow is thermally fully developed and subject to uniform heat flux boundary condition,

$$\frac{\partial T}{\partial x} = \frac{dT_m}{dx} = \text{constant}, \tag{69}$$

which means that the axial gradient of the temperature at any point equals the axial gradient of the mean temperature and is a constant.

This reduces Eq. (32) to

$$\rho c_p u \frac{dT_m}{dx} = k \frac{\partial^2 T}{\partial y^2} + \tau_{yx} \frac{\partial u}{\partial y} \tag{70}$$

There are two dependent variables in the above equation: T_m and T . To eliminate T_m, the above equation is integrated once across the gap between the plates:

$$\int_{-h}^{h} \rho c_p u \frac{dT_m}{dx} dy = \int_{-h}^{h} k \frac{\partial^2 T}{\partial y^2} dy + \int_{-h}^{h} \tau_{yx} \frac{\partial u}{\partial y} dy \tag{71}$$

Using the boundary conditions mentioned in Eq. (30), we obtain the following:

$$\frac{dT_m}{dx} \int_{-h}^{h} \rho c_p u dy = 2q + \int_{-h}^{h} \tau_{yx} \frac{\partial u}{\partial y} dy \tag{72}$$

$$\Rightarrow \frac{dT_m}{dx} = \frac{2q + \int_{-h}^{h} \tau_{yx} \frac{du}{dy}}{\int_{-h}^{h} \rho c_p u dy} \tag{73}$$

Insert this expression in Eq. (32) to obtain the following:

$$\rho c_p u \frac{2q + \int_{-h}^{h} \tau_{yx} \frac{du}{dy}}{\int_{-h}^{h} \rho c_p u dy} = k \frac{\partial^2 T}{\partial y^2} + \tau_{yx} \frac{\partial u}{\partial y} \tag{74}$$

The non-dimensional form of temperature is introduced as:

$$\theta = \frac{k(T - T_w)}{q(2h)} \tag{75}$$

Using the non-dimensional variables introduced in Eq. (74) and in Eq. (77), we obtain the non-dimensional form of the energy equation as:

$$\frac{d^2\theta}{dy'^2}+\tau'_{yx}\,Br\frac{du'}{dy'}-u'\times Br\frac{\int_{-1}^{1}\tau'_{yx}\frac{du'}{dy'}}{\int_{-1}^{1}u'dy'}=\frac{u'}{\int_{-1}^{1}u'dy'} \tag{76}$$

Where *Br* is the Brinkman number for viscoelastic fluids given by[1, 12-13]:

$$Br=\frac{\eta U^2}{(2h)q} \tag{79}$$

Using the expression for velocity profile in Eq. (25), we obtain the following:

$$\int_{-1}^{1}u'dy'=\frac{\varepsilon De^2G^3}{5}+\frac{G}{3}+2c'_2=\bar{A}\text{ (say)} \tag{80}$$

Similarly,

$$\int_{-1}^{1}\tau'_{yx}\frac{du'}{dy'}=\frac{2G^2}{3}+0.8\times\varepsilon De^2\times G^4 \tag{81}$$

Substituting the expressions for *u'*, $\frac{du'}{dy'}$ and those in Eq. (40) and Eq. (41), the energy equation reduces to:

$$\frac{d^2\theta}{dy'^2}+G^2Br\times y'^2+2\varepsilon De^2G^4\times Br\times y'^4-\bar{K}\left[\frac{\varepsilon De^2G^3}{2}y'^4+\frac{G}{2}y'^2+c'_2\right]-\frac{\left[\frac{\varepsilon De^2G^3}{2}y'^4+\frac{G}{2}y'^2+c'_2\right]}{\bar{A}}=0 \tag{82}$$

Where, $$\bar{K}=\frac{Br\left[\frac{2G^2}{3}+0.8\times\varepsilon De^2\times G^4\right]}{\bar{A}} \tag{83}$$

Integrating once with respect to *y'*

$$\frac{d\theta}{dy'}+G^2Br\times\frac{y'^3}{3}+\frac{2\varepsilon De^2G^4\times Br\times y'^5}{5}-\bar{K}\left[\frac{\varepsilon De^2G^3}{10}y'^5+\frac{G}{6}y'^3+c'_2\,y'\right]-\frac{\left[\frac{\varepsilon De^2G^3}{10}y'^5+\frac{G}{6}y'^3+c'_2\,y'\right]}{\bar{A}}+\bar{\bar{K}}=0 \tag{84}$$

Where $\bar{\bar{K}}$ is a constant of integration.

Since $$\left.\frac{d\theta}{dy'}\right|_{y'=0}=0\Rightarrow\bar{\bar{K}}=0 \tag{85}$$

Inserting the value of $\bar{\bar{K}}$ in Eq. (44) and integrating the resultant equation, the following is obtained:

$$\theta(y')+\left[\frac{2\varepsilon De^2G^4Br}{30}-\frac{\overline{K}\varepsilon De^2G^3}{60}-\frac{\varepsilon De^2G^3}{60\overline{\overline{A}}}\right]y'^6+$$
$$\left[\frac{G^2Br}{12}-(\overline{\overline{KG}})/24-\frac{G}{24\overline{\overline{A}}}\right]y'^4-\left[\frac{\overline{K}c'_2}{2}+\frac{c'_2}{2\overline{\overline{A}}}\right]y'^2+\overline{\overline{\overline{K}}}=0 \quad (86)$$

Here, $\overline{\overline{\overline{K}}}$ is the (second) constant of integration. To find its value, we use the boundary condition that $\theta(1)=0$

$$\Rightarrow \overline{\overline{\overline{K}}}=-\left[\frac{2\varepsilon De^2G^4Br}{30}-\frac{\overline{K}\varepsilon De^2G^3}{60}-\frac{\varepsilon De^2G^3}{60\overline{\overline{A}}}\right]-$$
$$\left[\frac{G^2Br}{12}-(\overline{\overline{KG}})/24-\frac{G}{24\overline{\overline{A}}}\right]+\left[\frac{\overline{K}c'_2}{2}+\frac{c'_2}{2\overline{\overline{A}}}\right] \quad (87)$$

To conclude, the expression for non-dimensional temperature θ is given by Eq. (46) where the value of constant of integration $\overline{\overline{\overline{K}}}$ is given by Eq. (87). To obtain the best profile of velocity number also we can consider the following formulations :

$$v_z\frac{\partial T}{\partial z}=\alpha_f\left\{\frac{\partial^2T}{\partial x^2}+\frac{\Pr}{c_p}\left(\frac{dv_z}{dx}\right)^2\right\} \quad (88)$$

$$\frac{\partial T}{\partial z}=\frac{dT_w}{dz}=\frac{dT_m}{dz}=\frac{q^n}{\rho_f v_{zm}c_p a}+\frac{\mu}{\rho_f v_{zm}c_p a}\int_0^a\left(\frac{dv_z}{dx}\right)^2dx \quad (89)$$

$$\theta=\frac{T_w-T}{(q''a/k)} \quad \text{and} \quad Br=\frac{\mu(a^2(-dp/dz)/\mu)^2}{aq''} \quad (90)$$

$$\frac{d^2\theta}{d\overline{x}^2}+\left(\frac{\overline{v}_z}{\overline{v}_{zm}}\right)\left[1+Br\int_0^1\left(\frac{d\overline{v}_z}{d\overline{x}}\right)^2d\overline{x}\right]=Br\left(\frac{d\overline{v}_z}{d\overline{x}}\right)^2 \quad (91)$$

$$\theta=Br\left[E_\theta-(\overline{x}-1)A_\theta-C_\theta\right]-\frac{1+BrJ}{\overline{v}_{zm}}\left[F_\theta-(\overline{x}-1)B_\theta-D_\theta\right] \quad (92)$$

$$J=\frac{1}{3}-\frac{8G_1\overline{\xi E}_s}{(ka)^2}\left[\ln\left(\frac{1+\eta e^{ka}}{1-\eta e^{ka}}\right)-\frac{1}{ka}\left\{\left[Li_2(\eta e^{ka})-Li_2(-\eta e^{ka})\right]-\left[Li_2(\eta)-Li_2(-\eta)\right]\right\}\right]$$
$$+\frac{32(G_1\overline{\xi})^2\overline{E}_s^2}{(ka)^3}\left[\frac{1}{1-(\eta e^{ka})^2}-\frac{1}{1-\eta^2}\right] \quad (93)$$

$$A_\theta=\frac{8G_1\overline{\xi E}_s}{(ka)^3}\left[Li_2(\eta)-Li_2(-\eta)\right]+\frac{32(G_1\overline{\xi})^2\overline{E}_s^2}{(ka)^3}\frac{1}{1-\eta^2} \quad (94)$$

$$B_\theta=\frac{4G_1\overline{\xi E}_s}{(ka)^3}\left[Li_2(\eta)-Li_2(-\eta)\right] \quad (95)$$

$$C_\theta=\frac{1}{12}-\frac{8G_1\overline{\xi E}_s}{(ka)^4}\left[ka\left\{Li_2(\eta e^{ka})-Li_2(-\eta e^{ka})\right\}-2\left\{Li_3(\eta e^{ka})-Li_3(-\eta e^{ka})\right\}\right]$$
$$+\frac{16(G_1\overline{\xi})^2\overline{E}_s^2}{(ka)^4}\ln\left(\frac{(\eta e^{ka})^2}{1-(\eta e^{ka})^2}\right) \quad (96)$$

$$D_\theta=\frac{5}{24}-\frac{G_1\overline{\xi}^2\overline{E}_s}{(ka)^2}+\frac{4G_1\overline{\xi E}_s}{(ka)^4}\left[Li_3(\eta e^{ka})-Li_3(-\eta e^{ka})\right] \quad (97)$$

$$E_\theta = \frac{\bar{x}^4}{12} - \frac{8G_1\bar{\xi}\bar{E}_s}{(ka)^4}\left[ka\bar{x}\left\{Li_2(\eta e^{ka\bar{x}}) - Li_2(-\eta e^{ka\bar{x}})\right\} - 2\left\{Li_3(\eta e^{ka\bar{x}}) - Li_3(-\eta e^{ka\bar{x}})\right\}\right] + \frac{16(G_1\bar{\xi})^2\bar{E}_s^2}{(ka)^4}\ln\left(\frac{(\eta e^{ka\bar{x}})^2}{1-(\eta e^{ka\bar{x}})^2}\right) \tag{98}$$

$$F_\theta = -\frac{\bar{x}^4}{24} + \frac{\bar{x}^2}{4} - \frac{G_1\bar{\xi}^2\bar{E}_s\bar{x}^2}{(ka)^2} + \frac{4G_1\bar{\xi}\bar{E}_s}{(ka)^4}\left[Li_3(\eta e^{ka\bar{x}}) - Li_3(-\eta e^{ka\bar{x}})\right] \tag{99}$$

$$Q = 2W\int_0^a v_z dx = 2Wav_{zm} \tag{100}$$

$$\bar{Q} = \frac{Q}{2aWv_0} = \bar{v}_{zm} \tag{101}$$

$$\bar{v}_{zm} = \int_0^1 \bar{v}_z d\bar{x} = \frac{1}{3} + \frac{4G_1\bar{\xi}\bar{E}_s}{(ka)^3}\left[\frac{(ka)\bar{\xi}}{2} + \left\{Li_2(\eta e^{ka}) - Li_2(-\eta e^{ka})\right\} - \left\{Li_2(\eta) - Li_2(-\eta)\right\}\right] \tag{102}$$

$$Q = \frac{2(-dp/dz)a^3W}{3\mu_0} \tag{103}$$

$$\bar{Q} = \frac{\mu}{3\mu_0} \tag{104}$$

$$\frac{\mu_0}{\mu} = \frac{1}{3\bar{v}_{zm}} = \frac{1}{3\bar{Q}} \tag{105}$$

$$f = \frac{2a(-dp/dx)}{\rho_f v_{zm}^2} \tag{106}$$

$$\mathrm{Re} = 4a\rho_f v_{zm}/\mu \tag{107}$$

$$f\,\mathrm{Re} = \frac{8}{\bar{v}_{zm}} = \frac{8}{\bar{Q}} \tag{108}$$

Which corresponds to the EDL layer of

$$\frac{d^2\bar{\psi}}{d\bar{x}^2} = (ka)^2\bar{\psi} \tag{109}$$

$$\bar{\psi} = \frac{\bar{\xi}\sinh(ka\bar{x})}{\sinh(ka)} \tag{110}$$

$$v_z = \frac{1}{2}(1-\bar{x}^2) - \frac{2\bar{E}_s G_1\bar{\xi}^2}{(ka)^2}\left[1 - \frac{\sinh(ka\bar{x})}{\sinh(ka)}\right] \tag{111}$$

$$\bar{I}_s = -2\int_0^1 \bar{v}_z\bar{\psi}d\bar{x} \tag{112}$$

$$\bar{I}_s = -2\alpha\left[\frac{1}{2}(I_4 - I_5) - \left(\frac{2\bar{E}_s G_1\bar{\xi}^2}{(ka)^2}\right)I_6 + \frac{2\bar{E}_s G_1\bar{\xi}^2}{(ka)^2\sinh(ka)}I_7\right] \tag{113}$$

$$I_4 = I_6 = \frac{\cosh(ka)-1}{ka} \tag{114}$$

$$I_5 = \left[\frac{1}{ka} + \frac{2}{(ka)^3}\right]\cosh(ka) - \frac{2}{(ka)^2}\sinh(ka) - \frac{2}{(ka)^3} \tag{115}$$

$$I_7 = \frac{1}{2}\left[\frac{1}{ka}\sinh(ka)\cosh(ka) - 1\right] \tag{116}$$

$$\alpha = \bar{\xi} / \sinh(ka) \quad (117)$$

$$\bar{E}_s = \frac{\dfrac{\alpha G_2 (ka)^2}{\bar{\xi}}(I_4 - I_5)}{1 + 4\alpha G_1 G_2 \bar{\xi}(I_6 - \dfrac{I_7}{\sinh(ka)})} \quad (118)$$

$$\bar{v}_{zm} = \frac{1}{3} - \frac{2G_1 \bar{\xi}^2 \bar{E}_s}{(ka)^2}\left[1 - \frac{\cosh(ka) - 1}{(ka)\sinh(ka)}\right] \quad (119)$$

$$f\,\mathrm{Re} = \frac{24}{1 - \dfrac{6G_1 \bar{\xi}^2 \bar{E}_s}{(ka)^2}\left[1 - \dfrac{\cosh(ka) - 1}{(ka)\sinh(ka)}\right]} = \frac{8}{\bar{Q}} = 24\left(\frac{\mu_0}{\mu}\right) \quad (120)$$

$$J = \frac{1}{3} - \frac{4\bar{E}_s G_1 \bar{\xi}^2}{(ka)^3 \sinh(ka)}\left[(ka)\sinh(ka) - \cosh(ka) + 1\right] + \frac{2\bar{E}_s^2 G_1^2 \bar{\xi}^4}{(ka)^3 \sinh^2(ka)}\left[\sinh(2ka) + 2ka\right] \quad (121)$$

$$A_\theta = \frac{4\bar{E}_s G_1 \bar{\xi}^2}{(ka)^3 \sinh(ka)} \quad (122)$$

$$B_\theta = \frac{2\bar{E}_s G_1 \bar{\xi}^2}{(ka)^3 \sinh(ka)} \quad (123)$$

$$C_\theta = \frac{1}{12} - \frac{4\bar{E}_s G_1 \bar{\xi}^2}{(ka)^4 \sinh(ka)}\left[(ka)\cosh(ka) - 2\sinh(ka)\right] + \frac{\bar{E}_s^2 G_1^2 \bar{\xi}^4}{2(ka)^4 \sinh^2(ka)}\left[\cosh(2ka) + 2(ka)^2\right] \quad (124)$$

$$D_\theta = \frac{5}{24} - \frac{\bar{E}_s G_1 \bar{\xi}^2}{(ka)^4}\left[(ka)^2 - 2\right] \quad (125)$$

$$E_\theta = \frac{\bar{x}^4}{12} - \frac{4\bar{E}_s G_1 \bar{\xi}^2}{(ka)^4 \sinh(ka)}\left[(ka\bar{x})\cosh(ka\bar{x}) - 2\sinh(ka\bar{x})\right] + \frac{\bar{E}_s^2 G_1^2 \bar{\xi}^4}{2(ka)^4 \sinh^2(ka)}\left[\cosh(2ka\bar{x}) + 2(ka)^2 \bar{x}^2\right] \quad (126)$$

$$F_\theta = \frac{\bar{x}^2}{24}(6 - \bar{x}^2) - \frac{\bar{E}_s G_1 \bar{\xi}^2}{(ka)^4}\left[(ka)^2 \bar{x}^2 - 2\frac{\sinh(ka\bar{x})}{\sinh(ka)}\right] \quad (127)$$

Withouth the slip wall effects we have :

$$T_m = \frac{2W\int_0^a T c_p \rho_f v_z dx}{\rho_f Q c_p} = \frac{\int_0^a T v_z dx}{a v_{zm}} = \frac{\int_0^1 T \bar{v}_z d\bar{x}}{\bar{v}_{zm}} \quad (128)$$

$$\theta_m = \int_0^1 \theta \left(\frac{\bar{v}_z}{\bar{v}_{zm}}\right)^2 d\bar{x} \quad (129)$$

$$\theta_m = \frac{Br\int_0^1 G_{v\theta} d\bar{x} - \left(\frac{1+BrJ}{\bar{v}_{zm}}\right)\int_0^1 H_{v\theta} d\bar{x}}{\bar{v}_{zm}} \tag{130}$$

$$G_{v\theta} = \bar{v}_z\left[E_\theta - (\bar{x}-1)A_\theta - C_\theta\right] \tag{131}$$

$$H_{v\theta} = \bar{v}_z\left[F_\theta - (\bar{x}-1)B_\theta - D_\theta\right] \tag{132}$$

$$Nu = \frac{hD_h}{k_f} = \frac{4aq''}{k_f(T_w - T_m)} = \frac{4}{\theta_m} \tag{133}$$

$$Nu = \frac{4\bar{v}_{zm}}{Br\int_0^1 G_{v\theta} d\bar{x} - \left(\frac{1+BrJ}{\bar{v}_{zm}}\right)\int_0^1 H_{v\theta} d\bar{x}} \tag{134}$$

The most relevant non-dimensional parameter in convection is the Nusselt number (*Nu*). It is a dimensionless form of the heat transfer coefficient near the wall (HTC):

$$Nu = \frac{HTC \times 4h}{k} \tag{135}$$

(for parallel plates, the hydraulic diameter is two times the gap).
For convection in a channel, the HTC is defined in terms of the mean temperature of the flow:

$HTC = q/(T_W - T_m)$ (136)

Writing in non-dimensional terms, the Nusselt number reduces to

$$Nu = \frac{-2}{\theta_m} \tag{137}$$

Here, $\theta_m \text{(mean-dimensionless-temperature)} = \frac{\int_{-1}^{1} u'\theta dy'}{\int_{-1}^{1} u' dy'} = \frac{\int_{-1}^{1} u'\theta dy'}{\bar{A}}$ (138)

The numerator of Eq. (51) is obtained by substituting the expressions for dimensionless velocity profiles and temperature profiles from Eq. (25) and Eq. (46) respectively.

$$\int_{-1}^{1} u'\theta dy' = \frac{AG}{5} + \frac{A\times\varepsilon De^2G^3}{7} + Ac'_2\times\left(\frac{2}{3}\right) - \frac{BG}{7} - \frac{B\times\varepsilon De^2G^3}{9} - B\times c'_2\left(\frac{2}{5}\right) - \frac{CG}{9} - \frac{C\times\varepsilon De^2G^3}{11} - \frac{2C\times c'_2}{7} - \frac{\bar{\bar{K}}\times G}{3} - \frac{\bar{\bar{K}}\times\varepsilon De^2G^3}{5} - 2\times\bar{\bar{K}}c'_2 \tag{139}$$

Here,

$$A = \left[\frac{\bar{K}c'_2}{2} + \frac{c'_2}{2\bar{\bar{A}}}\right]$$

$$B = \left[\frac{G^2Br}{12} - \overline{(KG)}/24 - \frac{G}{24\bar{\bar{A}}}\right] \tag{140}$$

$$C = \left[\frac{2\varepsilon De^2G^4Br}{30} - \frac{\bar{K}\varepsilon De^2G^3}{60} - \frac{\varepsilon De^2G^3}{60\bar{A}}\right]$$

Thus, substituting the values of *A*, *B*, *C* from Eq. (53) into Eq.(52), we calculate the numerator in Eq. (53) and get the expression for θ_m. This is then back-substituted in Eq. (50) to obtain the value of Nusselt number.

For the special case of Newtonian fluid with zero viscous dissipation, the Nusselt number reduces to 8.24 which shows a perfect match with the data in available literature [25]. For any non-Newtonian fluid, the rate of volumetric entropy generation rate is given by [15]:

$$\dot{S}'''_{gen} = \frac{k}{T^2}\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right] + \frac{\phi}{T} \tag{141}$$

Substituting the value of ϕ from Eq. (28)

$$\dot{S}'''_{gen} = \frac{k}{T^2}\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right] + \frac{\tau_{yx}}{T}\frac{du}{dy} \tag{142}$$

In non-dimensional form, the above equation reduces to:

$$N_s = \dot{S}'''_{gen}\frac{h^2}{k} = \frac{4\psi^2}{[\theta\psi+1]^2}\left[\frac{1+Br\int_{-1}^{1}\tau'_{yx}\frac{du'}{dy'}}{Pe\int_{-1}^{1}u'dy'}\right] + \frac{\psi^2}{[\theta\psi+1]^2}\left(\frac{\partial\theta}{\partial y'}\right)^2 + \frac{Br\times\psi}{(\theta\psi+1)}\tau'_{yx}\frac{du'}{dy'} \tag{143}$$

In the above expression, N_s is the entropy generation number , ψ is dimensionless heat flux equaling $\frac{(2h)q}{kT_w}$ and Pe is the Peclet number which has the expression $\frac{\rho c_p U(2h)}{k}$. For the case treated here, we have neglected axial conduction and that means $Pe \to \infty$. This reduces the expression for entropy generation to:

$$N_s = \frac{\psi^2}{[\theta\psi+1]^2}\left(\frac{\partial\theta}{\partial y'}\right)^2 + \frac{Br\times\psi}{(\theta\psi+1)}\tau'_{yx}\frac{du'}{dy'} \tag{144}$$

In this expression, the first term on the right denotes entropy generated due to heat transfer and the second term refers to the entropy generated due to fluid friction.

The average non-dimensional entropy generation rate for a cross-section is given by:

$$< N_s >= \frac{\int_{-1}^{1} N_s dy'}{2} \tag{145}$$

The expression for non-dimensional entropy generation in Eq. (144) conveys the information about the total entropy generation rate. But it does not show that out of the two entropy generation mechanisms namely, heat transfer and fluid friction, which one dominates. This information is provided to us by Bejan number *Be* [26]:

$$Be = \frac{EntropyGeneratedFromHeatTransfer}{TotalEntropyGenerated} \tag{146}$$

In the current case, this reduces to:

$$Be = \frac{\frac{\psi^2}{[\theta\psi+1]^2}\left(\frac{\partial\theta}{\partial y'}\right)^2}{N_s} \tag{147}$$

In the preceding sections, the problem statement has been presented and the mathematical equations governing the hydrodynamics, heat transfer and entropy generation characteristics of the physical model have been formulated. In this section, the results will be presented through suitable figures and tables. The reasons for the trends observed will be discussed threadbare and an attempt is made to gain more insight into the physics of slip flow of viscoelastic fluids.

An important parameter of interest in the study of viscoelastic fluids is the viscoelastic group - εDe^2 - the product of elongation parameter ε and De^2. Deborah number (*De*) can be interpreted as the ratio of elastic forces to viscous forces. A high value of *De* ensures that the fluid behaves like an elastic solid while for low values of *De*, its behavior will be close to viscous liquids. For this reason, the results in this paper are presented for two different values of εDe^2 -namely, 1 and 4. The results have been divided into three sub-sections: Navier's non- linear slip law, Hatzikiriakos slip law and asymptotic slip law. Due to symmetry, the results have been shown for only one half of the channel width. Finally a comparison has been made between the results for asymptotic slip law and Hatzikiriakos slip law.

For Navier's non- linear slip law, the values of the pertinent parameters are shown in Table 1. The value of Brinkman number has been taken as constant because the effect of *Br* on heat transfer characteristics of viscoelastic fluids have already been studied in literature [12-14].

The values of the non- dimensional pressure gradient (*G*), non-dimensional heat flux (ψ), slip coefficients k'_{nl} and *m* are consistent with those used elsewhere in literature [11]. For *m*, a value of 1 makes the Navier's slip law linear while that of 2 relates to Navier's non-linear slip law.

Near the core, the slope turns to zero for each profile. This can be explained as follows. The apparent viscosity of a non-Newtonian fluid is expressed as:

$$\mu' = \frac{\tau'_{xy}}{\frac{\partial u'}{\partial y'}} = \frac{1}{1 + 2\varepsilon De^2 \times \tau'_{xy}} \tag{148}$$

As the value of εDe^2 increases, the apparent viscosity decreases. The decrease in apparent viscosity is more pronounced near the walls where the shear stresses (τ'_{xy}) are higher. It means that the shear thinning behavior of fluid is enhanced with increase in εDe^2, leading to higher velocity gradients [27, 28]. In principle, since *De* is defined as the ratio of elastic forces and viscous forces in the fluid, a high value of *De* means that the viscosity of the fluid will be low and the shear thinning behavior will be enhanced.

A more mathematical approach to understand this is by looking at Eq. (25) which conveys that an increase in εDe^2 causes the velocity to increase. Since the velocity at the wall is fixed by the slip, the increase is felt mostly in core region. The increase in core velocity, without any change in velocity at the wall, causes the velocity gradients to rise.

CONCLUSION

This paper analyzes the effect of different slip laws on the heat transfer and entropy generation characteristics of viscoelastic fluids. The viscoelastic behavior was modeled through the linear version of simplified Phan-Thien-Tanner (s-PTT) model. Three different slip laws, including Navier's non-linear slip law, Hatzikiriakos slip law and asymptotic slip law, were used to capture the slip on the wall. The flow was hydrodynamically and thermally fully developed. A uniform heat flux was imposed on the walls and the effect of viscous dissipation was taken into account, while axial conduction was ignored. The governing equations for mass, momentum and energy conservation were solved analytically and exact expressions for velocity, stress, temperature, entropy generation, Bejan number and Nusselt number were obtained. The following conclusions can be drawn from this study.

1. For different values of slip, the velocity profile changes but the velocity gradient remains the same. Thus, a change in slip coefficients for symmetrical slip, will alter only the advection of the fluid momentum, but not its diffusion.
2. The shear thinning behavior of the fluid is enhanced with increase in the value of viscoelastic group εDe^2. This effect is more pronounced near the wall where the shear stress is higher.
3. An increase in slip coefficients/ slip velocity will lead to an increase in the temperature of the fluid because of an improved heat transfer rate at the wall.

4. The Nusselt number has a complex dependence on the viscoelastic group εDe^2, the pressure gradient and the slip coefficients. In general, an increase in the slip coefficients will enhance the value of *Nu* because of improved heat transfer rate; an increase in the value of εDe^2 will also lead to an increase in *Nu*. But for higher values of pressure gradient and slip velocity, an increase in εDe^2 will lead to lower *Nu*. This is attributed to the high flow rate at high pressure gradient and εDe^2, which brings down the average temperature.
5. The entropy generation rate decreases with increase in slip velocity because, thermodynamically, the friction at the wall is a source of irreversibility. A high value of slip suggests that there is less friction/ resistance to the flow at the wall and thus less irreversibility.
6. The Bejan number of the flow was found to be close to 1 for all the cases considered. Thus a researcher grappling with the problem of entropy generation minimization for viscoleastic fluids is advised to concentrate his efforts more towards the irreversibility due to heat transfer.
7. Bejan number is lowest in the vicinity of the wall because the velocity gradients are highest there.
8. Between the Hatzikiriakos and the asymptotic slip laws, the slip velocity modeled by Hatzikiriakos slip law is higher. This leads to higher value of Nusselt number and lower values of entropy generation rates for Hatzikiriakos slip law.

In future, the author intends to explore similar phenomena for exponential formulation of s-PTT model in an upcoming paper. Moreover, the entropy generation analysis covered here can lay the groundwork for an Entropy Generation Minimization (EGM) analysis [15] for further research in viscoelastic fluids.

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