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Research Article

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A important method for the probability limit theory of exchangeable random variables

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ABSTRACT

Limit theory mainly study independent random variables, but in many practical problems, samples are not independent, or the function of independent sample is not independent, or the verification of independent is more difficult. So the concept of dependent random variables in probability and statistics is mentioned. Exchangeable random variables is a major type of dependent random variable. By using reverse martingale, censored and other methods, in certain relevant conditions, we extend some conclusions to the exchangeable random variables, and obtain several conclusions of the convergence of exchangeable random variables.

Key words: limit theory; exchangeable random variables; reverse martingale

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INTRODUCTION

If the joint distribution of $X_1, X_2, \cdots, X_n$ is permutation invariant, for each permutation π of $1, 2, \cdots, n$, the joint distribution of $X_1, X_2, \cdots, X_n$ is the same of the joint distribution of $X_{\pi(1)}, X_{\pi(2)}, \cdots, X_{\pi(n)}$, so the finite random variable sequence $X_1, X_2, \cdots, X_n$ is exchangeable. Obviously, the independent and identically distributed random variable sequence is the simplest exchangeable random variable sequence. The concept of exchangeability was first proposed by De Finett[1] in 1930, people using the De Finetti theorem has made some results (see [2]-[3]). In this paper we extend the results of Katz and Baum theorem in the condition of independent and identically distributed random variable sequence to the results of Katz and Baum theorem in the condition of exchangeable random variables. We obtain the Katz and Baum theorem for specific forms of expression in the case of exchangeable random variables.

PRELIMINARY CONCEPTS

Definition [4]: when on the $[0, \infty)$ the positive function $l(x)(x \to \infty)$ is slowly varying function, as for all $c > 0$, $\lim\limits_{x\to\infty} \frac{l(cx)}{l(x)} = 1$. About slowly varying function with the following properties:if it is slowly varying function when $l(x)(x \to \infty)$, so

(1) $\lim\limits_{x\to\infty} \frac{l(tx)}{l(x)} = 1,\ \forall t > 0,\ \lim\limits_{x\to\infty} \frac{l(x+u)}{l(x)} = 1,\ \forall u \ge 0$;

(2) $\lim\limits_{k\to\infty} \sup\limits_{2^k \le x < 2^{k+1}} \frac{l(x)}{l(2^k)} = \lim\limits_{k\to\infty} \inf\limits_{2^k \le x < 2^{k+1}} \frac{l(x)}{l(2^k)} = 1$;

(3) $\lim\limits_{x\to\infty} x^{\delta} l(x) = \infty,\ \forall \delta > 0,\ \lim\limits_{x\to\infty} x^{-\delta} l(x) = 0$.

Lemma 1 Suppose $\{X_n; n \ge 1\}$ are exchangeable random variables, and satisfy

$Cov(f_1(X_1), f_2(X_2)) \le 0$

$\forall m \ge 2,\ A_1, A_2, \cdots A_m$ is $\{1,2,\cdots,n\}$ twenty-two disjoint non-empty set,if $f_i, i = 1,2,\cdots,m$ each argument is a function of both the non-drop (non-liters,so

(1) if $f_i \ge 0, i = 1,2,\cdots,m$,we obtain

$$E\left(\prod_{i=1}^{n} f_i\left(X_j, j \in A_i\right)\right) \le \prod_{i=1}^{n} Ef_i\left(X_j, j \in A_i\right)$$

(2) $\forall\ x_i \in R, i = 1,2,\cdots,m,$

$$P\left(X_1 < x_1, \cdots X_m < x_m\right) \le \prod_{i=1}^{n} P\left(X_i < x_i\right)$$

Lemma 2 Suppose $\{X_n; n \ge 1\}$ are exchangeable random variables,

$Cov(f_1(X_1), f_2(X_2)) \le 0$

$EX_n = 0, |X_n| \le b_n \quad a.s.\ (n = 1,2,\cdots),\ t > 0$ and $t \max_{1 \le i \le n} b_i \le 1$,so $\forall u > 0$,we obtain

$$P\left(\left|\sum_{i=1}^{n} X_i\right| \ge u\right) \le 2\exp\left\{-tu + t^2 \sum_{i=1}^{n} EX_i^2\right\}$$

Proof: because $Y_n \,\square\, \sum_{k=0}^{n} \frac{(tX_i)^k}{k!} \to e^{tX_i}$, $|tX_i| \le 1\ a.s.$ we obtain $|Y_n| \le e\ a.s.$.so

By Lebesgue Theorem control convergence,we obtain

$$E\left(e^{tX_i}\right) = E\left(\lim_{n\to\infty} \sum_{k=0}^{n} \frac{(tX_i)^k}{k!}\right) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{E(tX_i)^k}{k!}$$

$$= \sum_{k=0}^{n} \frac{E(tX_i)^k}{k!} \le 1 + E(tX_i)^2 \sum_{k=2}^{\infty} \frac{1}{k!}$$

$$\le 1 + t^2 EX_i^2 \le e^{t^2 EX_i^2}$$

By Markov inequality and lemma1(1),we obtain $\forall u > 0,\ t > 0$,有

$$P\left(\sum_{i=1}^{n} X_i > u\right) = P\left(e^{t\sum_{i=1}^{n} X_i} > e^{tu}\right) \le e^{-tu} Ee^{t\sum_{i=1}^{n} X_i}$$

$$\le e^{-tu} \prod_{i=1}^{n} Ee^{tX_i} \le e^{-tu} \prod_{i=1}^{n} e^{t^2 EX_i^2}$$

$$= \exp\left(-tu + t^2 \sum_{i=1}^{n} EX_i^2\right)$$

We use $-X_i$ instead of X_i

$$P\left(\sum_{i=1}^{n} (-X_i) > u\right) = P\left(\sum_{i=1}^{n} (X_i) < -u\right) \le \exp\left(-tu + t^2 \sum_{i=1}^{n} EX_i^2\right)$$

$$P\left(\left|\sum_{i=1}^{n} X_i\right| \ge u\right) = P\left(\sum_{i=1}^{n} X_i > u\right) + P\left(\sum_{i=1}^{n} X_i < -u\right)$$

$$\le 2\exp\left(-tu + t^2 \sum_{i=1}^{n} EX_i^2\right)$$

THE MAIN CONCLUSION

Theorem: Suppose $\{X_n; n \ge 1\}$ are exchangeable random variables, and the P moments is exist,when $0 < p \le 1$ it is an arbitrary random sequence, when $p > 1$ it is exchangeable random variables of zero mean, when $0 < p \le 3$,

$$E\left(\max_{1\le j\le n}|S_j|^p\right) \le c\sum_{i=1}^{n} E|X_i|^p$$

when $p > 3$,

$$E\left(\max_{1\le j\le n}|S_j|^p\right) \le c\left\{\sum_{j=1}^{n} E|X_j|^p + \left(\sum_{j=1}^{n} EX_j^{\ 2}\right)^{p/3}\right\} \qquad (1)$$

Proof: when $0 < p \le 1$,by Jensen inequality, we obtain

$$E\left(\max_{1\le j\le n}|S_j|^p\right) = E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \le E\left(\max_{1\le j\le n}\left(\sum_{i=1}^{j}|X_i|\right)^p\right)$$

$$\le E\left(\sum_{i=1}^{j}|X_i|\right)^p \le \sum_{i=1}^{j} E|X_i|^p$$

so $E\left(\max_{1\le j\le n}|S_j|^p\right) \le c\sum_{i=1}^{n} E|X_i|^p$ is right.

when $p > 1$,it is easy to prove the following inequality:

$$|1+t|^p \le 1 + pt + 2^{3-p}|t|^p,\ \forall t \in R,\ 0 < p \le 3$$

$$|1+t|^p \le 1 + pt + 2^p p^2 t^2 + 2^p|t|^p,\ \forall t \in R,\ p > 3$$

suppose $t = x/y$,by the above formula $\forall x,\ y \in R$, we obtain

$$|x+y|^p = |y|^p\left(1+\frac{x}{y}\right)^p \le |y|^p\left(1 + p\frac{x}{y} + 2^{3-p}\left|\frac{x}{y}\right|^p\right)$$

$$\le 2^{2-p}|x|^p + px|y|^{p-1}\operatorname{sgn} y,\ 0 < p \le 3 \qquad (2)$$

$$|x+y|^p \le |x|^p + |y|^p + px\operatorname{sgn} y + 2^p p^2|y|^{p-2},\ p > 3$$

$$\max_{1\le j\le n}|S_j|^p = \left\{\max_{1\le j\le n}|S_j|\right\}^p$$

$$\le \{\max(0, S_1, \cdots, S_n)\}^p + \{\max(0, -S_1, \cdots, -S_n)\}^p$$

$$\le 2^{p-1}\{\max(0, S_1, \cdots, S_n)\}^p + 2^{p-1}\{\max(0, -S_1, \cdots, -S_n)\}^p$$

$$\le 2^{p-1}\left|\max_{1\le j\le n} S_j\right|^p + 2^{p-1}\left|\max_{1\le j\le n}(-S_j)\right|^p \qquad (3)$$

because $0 \le j \le n-1$,we note

$$M_{j,n} = \max(X_{j+1}, X_{j+1} + X_{j+2}, \cdots, X_{j+1} + X_{j+2} +, \cdots, + X_n)$$

$$0 \le \tilde{M}_{j,n} = \max(0, X_{j+1}, X_{j+1} + X_{j+2}, \cdots, X_{j+1} + X_{j+2} +, \cdots, + X_n)$$

First we prove $E\left(\max_{1\le j\le n}|S_j|^p\right) \le c\sum_{i=1}^{n} E|X_i|^p$,from $0 < p \le 3$,we use the formula of (2), we obtain

$$\left|\max_{1\le j\le n} S_j\right|^p = |M_{0,n}|^p + |X_1 + \tilde{M}_{1,n}|^p \le 2^{2-p}|X_1|^p + \tilde{M}_{1,n}^{\,p} + pX_1\tilde{M}_{1,n}^{\,p-1}$$

$$\le 2^{2-p}|X_1|^p + |M_{1,n}|^p + pX_1\tilde{M}_{1,n}^{\,p-1}$$

$$\vdots$$

$$\le 2^{2-p}\sum_{i=1}^{n}\left|X_i\right|^p + p\sum_{i=1}^{n-1}X_j\tilde{M}_{j,n}^{p-1} \qquad (4)$$

because X_j and $\tilde{M}_{j,n}^{p-1}$ are non-drop each on the corresponding variable, they also satisfy $Cov\left(f_1\left(X_j\right), f_2\left(\tilde{M}_{j,n}^{p-1}\right)\right)\le 0$, and we notice $EX_i = 0$,so

$E\left(X_j\tilde{M}_{j,n}^{p-1}\right)\le EX_jE\tilde{M}_{j,n}^{p-1} = 0,\ 0\le j\le n-1$

by the above formula and (4), we obtain

$$E\left(\left|\max_{1\le j\le n} S_j\right|^p\right)\le 2^{2-p}\sum_{i=1}^{n}E\left|X_i\right|^p$$

In the course of discussion, we use $-X_i$ instead of X_i ,we also obtain

$$E\left(\left|\max_{1\le j\le n}\left(-S_j\right)\right|^p\right)\le 2^{2-p}\sum_{i=1}^{n}E\left|X_i\right|^p \qquad (5)$$

form(3),(4) ,(5),we know $E\left(\max_{1\le j\le n}\left|S_j\right|^p\right)\le c\sum_{i=1}^{n}E\left|X_i\right|^p$ is right.

Let us prove the case when $p>3$,

In order to improve(1),we first prove

$$E\left(\max_{1\le j\le n}\left|S_j\right|^p\right)\le cE\left(\sum_{j=1}^{n}X_j^{\ 2}\right)^{p/3}$$

We note $c_1 = 3^p,\ c_2 = 3^p p^3$ .because $p>3$, $\left|x+y\right|^p\le\left|x\right|^p+\left|y\right|^p + px\operatorname{sgn} y + 3^p p^3\left|y\right|^{p-3}$,we obtain

$$\left|\max_{1\le j\le n} S_j\right|^p = \left|M_{0,n}\right|^p = \left|X_1+\tilde{M}_{1,n}\right|^p$$

$$\le c_1\left|X_1\right|^p + \tilde{M}_{1,n}^{p} + pX_1\tilde{M}_{1,n}^{p-1} + c_2X_1^2\tilde{M}_{1,n}^{p-2}$$

$$\le c_1\left|X_1\right|^p + \left|M_{1,n}\right|^p + pX_1\tilde{M}_{1,n}^{p-1} + c_2X_1^2\tilde{M}_{1,n}^{p-2}$$

$$\vdots$$

$$\le c_1\sum_{i=1}^{n}\left|X_i\right|^p + p\sum_{j=1}^{n-1}X_j\tilde{M}_{j,n}^{p-1} + c_2\sum_{j=1}^{n-1}X_j^2\tilde{M}_{j,n}^{p-2}$$

obviously,对 $0\le j\le n-1$,we obtain

$$\tilde{M}_{j,n}^{p-2} = \left(\max\left(S_j, S_j+X_{j+1}, S_j+X_{j+1}+X_{j+2},\cdots,S_j+X_{j+1}+,\cdots,+X_n\right)-S_j\right)^{p-3}$$

$$\le c\left(\left|\max_{j\le k\le n} S_k\right|^{p-3} + \left|S_j\right|^{p-3}\right)\le c\max_{1\le j\le n}\left|S_j\right|^{p-3}$$

from $E\left(X_j\tilde{M}_{j,n}^{p-1}\right)\le EX_jE\tilde{M}_{j,n}^{p-1} = 0,\ 0\le j\le n-1$ and all of the above, we obtain

$$E\left(\left|\max_{1\le j\le n} S_j\right|^p\right)\le c\left\{\sum_{i=1}^{n}E\left|X_i\right|^p + E\left(\sum_{i=1}^{n}X_i^{\ 2}\max_{1\le j\le n}\left|S_j\right|^{p/3}\right)\right\} \qquad (6)$$

The result also is right of $E\left|\max_{1\le j\le n}\left(-S_j\right)\right|^p$. So from (3)and (6), we use the Inequality of Hölder,we obtain

$$E\left(\left|\max_{1\le j\le n} S_j\right|^p\right)\le c\left\{\sum_{i=1}^{n}E\left|X_i\right|^p + E\left(\sum_{i=1}^{n}X_i^{\ 2}\max_{1\le j\le n}\left|S_j\right|^{p/3}\right)\right\}$$

$$\le c\left\{\sum_{i=1}^{n}E\left|X_i\right|^p + \left\{E\left(\sum_{i=1}^{n}X_i^{\ 2}\right)^{p/3}\right\}^{3/p}\left(E\max_{1\le j\le n}\left|S_j\right|^p\right)^{(p-2)/p}\right\}$$

$$\le c\sum_{i=1}^{n}E\left|X_i\right|^p+\frac{3}{p}c^{p/3}E\left(\sum_{i=1}^{n}X_i^{\,2}\right)^{p/3}+\frac{p-2}{p}\left(E\max_{1\le j\le n}\left|S_j\right|^p\right)$$

We move the formula $E\max_{1\le j\le n}\left|S_j\right|^p$ above from right to left, we obtain

$$E\left(\max_{1\le j\le n}\left|S_j\right|^p\right)\le c\left\{\sum_{i=1}^{n}E\left|X_i\right|^p+E\left(\left(\sum_{i=1}^{n}X_i^{\,2}\right)^{p/3}\right)\right\}$$

from $p>3$, so $0<3/p<1$, so

$$\sum_{i=1}^{n}E\left|X_i\right|^p=E\left(\sum_{i=1}^{n}\left|X_i\right|^p\right)=E\left\{\left(\sum_{i=1}^{n}\left|X_i\right|^p\right)^{3/p}\right\}^{p/3}\le E\left(\sum_{i=1}^{n}X_i^{\,2}\right)^{p/3}$$

We obtain

$$E\left(\max_{1\le j\le n}\left|S_j\right|^p\right)\le cE\left(\sum_{j=1}^{n}X_j^{\,2}\right)^{p/3}$$

Finally we prove the formula of $E\left(\max_{1\le j\le n}\left|S_j\right|^p\right)\le c\left\{\sum_{j=1}^{n}E\left|X_j\right|^p+\left(\sum_{j=1}^{n}EX_j^{\,2}\right)^{p/3}\right\}$

We note $X_i^+=X_iI_{(X_i\ge 0)}$, $X_i^-=-X_iI_{(X_i<0)}$, and suppose

$\xi_i=\left(X_i^+\right)^2-E\left(X_i^+\right)^2$, $\eta_i=\left(X_i^-\right)^2-E\left(X_i^-\right)^2$

obviously, $E\left(X_i^+\right)^2\le EX_i^2$, $E\left(X_i^-\right)^2\le EX_i^2$, from(1), we obtain

$$E\left(\left|\max_{1\le j\le n}S_j\right|^p\right)\le cE\left(\sum_{i=1}^{n}X_i^2\right)^{p/3}=cE\left(\sum_{i=1}^{n}\left(X_i^+-X_i^-\right)^2\right)^{p/3}$$

$$\le c\left\{E\left(\sum_{i=1}^{n}\left(X_i^+\right)^2\right)^{p/3}+E\left(\sum_{i=1}^{n}\left(X_i^-\right)^2\right)^{p/3}\right\}$$

$$\le c\left\{E\left(\sum_{i=1}^{n}\xi_i+\sum_{i=1}^{n}E\left(X_i^+\right)^2\right)^{p/3}+E\left(\sum_{i=1}^{n}\eta_i+\sum_{i=1}^{n}E\left(X_i^-\right)^2\right)^{p/3}\right\}$$

$$\le c\left\{E\left(\sum_{i=1}^{n}\xi_i\right)^{p/3}+E\left(\sum_{i=1}^{n}\eta_i\right)^{p/3}+\left(\sum_{i=1}^{n}EX_i^2\right)^{p/3}\right\} \qquad (7)$$

Here we use (7) ,and use the proof of induction on p ,so we prove(1). First we notice that ξ_i is non-decreasing function of X_i, η_i is non-increasing function of X_i, $\{\xi_i;i\ge 1\}$ and $\{\eta_i;i\ge 1\}$ Still is the exchangeable random variables with zero mean,And satisfies

$$Cov\left(f_1(\xi_1),f_2(\xi_2)\right)\le 0,\ Cov\left(f_1(\eta_1),f_2(\eta_2)\right)\le 0$$

When $3<p\le 3^2$, $1<p/3\le 3$, from the above, we obtain

$$E\left(\max_{1\le j\le n}\left|S_j\right|^p\right)\le c\sum_{i=1}^{n}E\left|X_i\right|^p$$

$$E\left(\sum_{i=1}^{n}\xi_i\right)^{p/3}\le c\sum_{i=1}^{n}E\left|\xi_i\right|^{p/3}\le c\left\{\sum_{i=1}^{n}E\left(X_i^+\right)^p+\sum_{i=1}^{n}\left(E\left(X_i^+\right)^2\right)^{p/3}\right\}$$

$$\le c\left\{\sum_{i=1}^{n}E\left|X_i\right|^p+\left(\sum_{i=1}^{n}EX_i^{\,2}\right)^{p/3}\right\} \qquad (8)$$

The formula $E\left(\sum_{i=1}^{n}\eta_i\right)^{p/3}$ also is right ,from the formula(8)we obtain (1) is right.

We suppose that when $3^{k-1} < p \le 3^k$ ($k \ge 2$) is right,so (8) is right,when $3^k < p \le 3^{k+1}$,(8) also is right. Use the induction hypothesis

$$E\left(\sum_{i=1}^{n}\xi_i\right)^{p/3} \le c\left\{\sum_{i=1}^{n}E|\xi_i|^{p/3} + \left(\sum_{i=1}^{n}E\xi_i^2\right)^{p/4}\right\}$$

$$\le c\left\{\sum_{i=1}^{n}E|X_i|^{p} + \left(\sum_{i=1}^{n}EX_i^2\right)^{p/3} + \left(\sum_{i=1}^{n}EX_i^2\right)^{p/9}\right\}$$

when $p > 4$, from the inequality of Hölder,we obtain

$$\sum_{i=1}^{n}EX_i^4 = \sum_{i=1}^{n}E\left\{\left(X_i^2\right)^{(p-4)/(p-2)}\left(|X_i|^p\right)^{3/(p-2)}\right\}$$

$$\le \sum_{i=1}^{n}\left(EX_i^2\right)^{(p-4)/(p-2)}\left(E|X_i|^p\right)^{3/(p-2)}$$

$$\le \left(\sum_{i=1}^{n}EX_i^2\right)^{(p-4)/(p-2)}\left(\sum_{i=1}^{n}E|X_i|^p\right)^{2/(p-2)}$$

From the real number of Hölder inequality, we know

$$\left(\sum_{i=1}^{n}EX_i^4\right)^{p/4} \le \left(\sum_{i=1}^{n}EX_i^2\right)^{p(p-4)/4(p-2)}\left(\sum_{i=1}^{n}E|X_i|^p\right)^{p/3(p-2)}$$

$$\le \frac{p-4}{2(p-2)}\left(\sum_{i=1}^{n}EX_i^2\right)^{p/3} + \frac{p}{2(p-2)}\sum_{i=1}^{n}E|X_i|^p \qquad (9)$$

from(8) and (9), we obtain

$$E\left(\sum_{i=1}^{n}\xi_i\right)^{p/3} \le c\left\{\sum_{i=1}^{n}E|X_i|^p + \left(\sum_{i=1}^{n}EX_i^2\right)^{p/3}\right\} \qquad (10)$$

We also obtain

$$E\left(\sum_{i=1}^{n}\eta_i\right)^{p/3} \le c\left\{\sum_{i=1}^{n}E|X_i|^p + \left(\sum_{i=1}^{n}EX_i^2\right)^{p/3}\right\} \qquad (11)$$

from(7),(10)and (11),we know (1)is right.

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