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A algorithm based on the PSS splitting for saddle-point problems

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ABSTRACT

We present a algorithm based on the PSS splitting for the saddle-point problems in this paper, and analysis the convergence of the new method. The results supported by numerical experiments show that the new method outperform the existing classical iterative methods.

Key words: Saddle-Point problem; PSS splitting; Convergence

INTRODUCTION

The special linear system is called Saddle-Point problems as follows,

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^* \\ -\boldsymbol{B} & \boldsymbol{O} \end{pmatrix}\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{pmatrix} \text{ or } \mathsf{A} \quad X = b, \tag{1.1}$$

where $\mathsf{A} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^* \\ -\boldsymbol{B} & \boldsymbol{O} \end{pmatrix}$, $\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{pmatrix}$, and $\boldsymbol{A} \in \boldsymbol{C}$ is a non-Hermitian positive definite matrix, $\boldsymbol{B} \in \boldsymbol{C}^{m\times n}$ with full row rank, $\boldsymbol{x}, \boldsymbol{f} \in \boldsymbol{C}$, $\boldsymbol{y}, \boldsymbol{g} \in \boldsymbol{C}$, $\boldsymbol{B}^*$ denotes the conjugate transpose of $\boldsymbol{B}$. It is widely used in many engineering problems such as hydrodynamics(Stokes problems), Least squares problem, optimization problem, elliptic partial differential equation of mixed finite element discretization and image processing etc.[1-5] The Saddle-Point problem has several distinctive features, firstly, the coefficient matrix is symmetric indefinite matrices generally, besides, it often contains singular matrix in diagonal block. Because of these two characteristics, saddle-point problem is often very sick. Due to the inherent characteristics of it, many well-known iterative methods are difficult to use directly, such as SOR, Gauss-Seidel, PCG algorithms etc. In 1958, Uzawa[6] Proposed the Uzawa method for solving the quadratic optimization problems effectively in economics. This method is based on the matrix splitting and it is simple, easy to realize by computer, so it is popular for people. However, it also has its drawbacks, namely the iterations are required to calculate the inverse matrix of $\boldsymbol{A}$, it is very difficult to realize for the large systems of linear equations. There for it has important theoretical and practical significance for seeking rapid and efficient iterative method, and thus the research become one of the hot issues of many scholars. After Uzawa researched the problem, Elman and other scholars[1]proposed and analyzed pretreatment Uzawa method and inexact Uzawa method. Gulob etc.[7] proposed the SOR-Like method based on the SOR method in 2001. And this method was often applied to the pretreatment of Krylov subspace iteration. Benzi etc. [8]proposed the parameter iterative method based on the HSS method in 2004. Z. Z. Bai etc.[9] propose the classical HSS algorithm in 2003, and it converged for positive parameter arbitrary using the alternating iterative method and matrix division. This aroused the attention of scholars greatly. Then, Z. Z. Bai etc.[10] proposed the accelerate Hss(AHSS) method based on the HSS method and special structure of saddle-point problem. H.K. Pang[11]proposed the pretreatment of non symmetric saddle-point problem in 2008. There are still many scholars researching to solve the saddle–point problem.

In this paper, a generalized local method based on the PSS iterative method for the saddle-point problem is proposed.

THE ALGORITHM FOR THE SADDLE-POINT PROBLEM

For the following positive definite equation

$$\boldsymbol{Ax} = \boldsymbol{b}\,, \tag{2.1}$$

where $\boldsymbol{A} \in \boldsymbol{C}^{n\times n}$ and $\boldsymbol{x,b} \in \boldsymbol{C}^{n}$ .The coeffient matrix could be splited into

$$\boldsymbol{A} = \boldsymbol{P} + \boldsymbol{S}\,, \tag{2.2}$$

where $\boldsymbol{P}$ is positive definite matrix and S is Skew-Hermitian matrix. Using the following PSS iterative method, namely given an initial vector $\boldsymbol{x}^{(0)} \in \boldsymbol{C}^{n}$, by solving a lineare system of equations to calculate $\boldsymbol{x}^{(k+1)}$ until the iterative sequence $\left\{\boldsymbol{x}^{(k)}\right\}_{k=0}^{\infty}$ converges, we have

$$\begin{cases} (\alpha\boldsymbol{I} + \boldsymbol{P})\boldsymbol{x}^{(k+\frac{1}{2})} = (\alpha\boldsymbol{I} - \boldsymbol{S})\boldsymbol{x}^{(k)} + \boldsymbol{b} \\ (\alpha\boldsymbol{I} + \boldsymbol{S})\boldsymbol{x}^{(k+1)} = (\alpha\boldsymbol{I} - \boldsymbol{P})\boldsymbol{x}^{(k+\frac{1}{2})} + \boldsymbol{b} \end{cases},$$

where α is a given positive constant and $\boldsymbol{I}$ is identity matrix.

Lemma 2.1[12] If $\boldsymbol{A} \in \boldsymbol{C}^{n\times n}$ is a positive matrix, and $\boldsymbol{A} = \boldsymbol{P} + \boldsymbol{S}$, the iterative matrix of PSS method is

$$\boldsymbol{M}(\alpha) = (\alpha\boldsymbol{I} + \boldsymbol{S})^{-1}(\alpha\boldsymbol{I} - \boldsymbol{P})(\alpha\boldsymbol{I} + \boldsymbol{P})^{-1}(\alpha\boldsymbol{I} - \boldsymbol{S})$$

Let

$$\boldsymbol{V}(\alpha) = (\alpha\boldsymbol{I} - \boldsymbol{P})(\alpha\boldsymbol{I} + \boldsymbol{P})^{-1}$$

then the upper bound for the spectral radius $\rho(\boldsymbol{M}(\alpha))$ of $\boldsymbol{M}(\alpha)$ is $\left\|\boldsymbol{V}(\alpha)\right\|_2$, and

$$\rho(\boldsymbol{M}(\alpha)) \le \left\|\boldsymbol{V}(\alpha)\right\|_2 < 1, \forall\alpha > 0$$

namely the PSS method convergeces the unique solution of $\boldsymbol{Ax} = \boldsymbol{b}$ for any given initial vector.

Lemma 2.2[8] Let $\mathsf{A} \in \boldsymbol{C}^{(m+n)\times(m+n)}$ be the coefficient matrix in (1.1), $\boldsymbol{A}$ be a Skew-Hermitian positive matrix, $\boldsymbol{B}$ with full row rank, $\rho(\mathsf{A})$ be the spectral radius of A , and $\lambda \in \rho(\mathsf{A})$ be a eigenvalue of A , then

1. A is nonsingular, and $\det(\mathsf{A}) > 0$;

2. A is semidefinite, that is for any $\boldsymbol{v} \in \boldsymbol{C}^{m+n}(\boldsymbol{v} \ne \boldsymbol{0})$, we have $\mathrm{Re}(\boldsymbol{v}^*\mathsf{A}\,\boldsymbol{v}) \ge 0$;

3. A is positively stable, that is for any $\lambda \in \rho(\mathsf{A})$, we have $Re(\lambda) > 0$.

Lemma 2.3[13] If $\boldsymbol{S}$ is skew-Hermite matrix, $i \cdot \boldsymbol{S}$ (i is imaginary unit) is Hermite matrix, and $\boldsymbol{u}^*\boldsymbol{Su}$ is pure imaginary or 0 for any $\boldsymbol{u} \in \boldsymbol{C}^{n}$.

Lemma 2.4[14] The modulus of the root of $\lambda^2 + \phi\lambda + \varphi = \boldsymbol{0}$ is less than 1 if and only if

$$\left|\phi - \bar{\phi}\varphi\right| + \left|\varphi\right|^2 < \boldsymbol{1},$$

where $\bar{\phi}$ is the conjugate of ϕ .

For the coefficient matrix A (See its properties in Lemma 2.2) of saddle-point problems in (1.1), we consider the special division as follows

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^* \\ -\boldsymbol{B} & \boldsymbol{O} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Q_1} + \boldsymbol{P} & \boldsymbol{O} \\ -\boldsymbol{B} & \boldsymbol{Q}_2 \end{pmatrix} - \begin{pmatrix} \boldsymbol{Q}_1 - \boldsymbol{S} & -\boldsymbol{B}^* \\ \boldsymbol{O} & \boldsymbol{Q}_2 \end{pmatrix}, \tag{2.3}$$

where if the Hermite part

$$\frac{1}{2}(\boldsymbol{P}^* + \boldsymbol{P}) = \frac{1}{2}(\boldsymbol{A}^* + \boldsymbol{A})$$

is positive definite, $\boldsymbol{P} \in \boldsymbol{C}^{n\times n}$ is positive definite, and $\boldsymbol{S} \in \boldsymbol{C}^{n\times n}$ is skew-Hermite matrix, $\boldsymbol{B}$ has full row-rank, that is $rank(\boldsymbol{B}) = m$. $\boldsymbol{Q}_1 \in \boldsymbol{C}^{n\times n}$ is Hermite matrix, $\boldsymbol{Q}_2 \in \boldsymbol{C}^{m\times m}$ is Hermite positive matrix, then we have

$$\begin{pmatrix} \boldsymbol{Q_1} + \boldsymbol{P} & \boldsymbol{O} \\ -\boldsymbol{B} & \boldsymbol{Q}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}^{(n+1)} \\ \boldsymbol{y}^{(n+1)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Q}_1 - \boldsymbol{S} & -\boldsymbol{B}^* \\ \boldsymbol{O} & \boldsymbol{Q}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}^{(n)} \\ \boldsymbol{y}^{(n)} \end{pmatrix} + \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{pmatrix}, \tag{2.4}$$

(2.4) can be written as

$$\begin{cases} \boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)} + (\boldsymbol{Q}_1 + \boldsymbol{P})^{-1}\left(\boldsymbol{f} - \boldsymbol{A}\boldsymbol{x}^{(n)} - \boldsymbol{B}^*\boldsymbol{y}^{(n)}\right) \\ \boldsymbol{y}^{(n+1)} = \boldsymbol{y}^{(n)} + \boldsymbol{Q}_2^{-1}\left(\boldsymbol{B}\boldsymbol{x}^{(n+1)} + \boldsymbol{g}\right) \end{cases}, \tag{2.5}$$

that is the new iterative method.

The iterative matrix of above iterative scheme is as follows

$$\tilde{\boldsymbol{M}} = \begin{pmatrix} \boldsymbol{Q_1} + \boldsymbol{P} & \boldsymbol{O} \\ -\boldsymbol{B} & \boldsymbol{Q}_2 \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{Q}_1 - \boldsymbol{S} & -\boldsymbol{B}^* \\ \boldsymbol{O} & \boldsymbol{Q}_2 \end{pmatrix}. \tag{2.6}$$

Let $\rho(\tilde{\boldsymbol{M}})$ be the spectral radius of $\tilde{\boldsymbol{M}}$,then if and only if

$$\rho(\tilde{\boldsymbol{M}}) < 1,$$

The iterative scheme is convergent, see [9-10,14-16].

Let λ be a eigenvalue of $\tilde{\boldsymbol{M}}$, $\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix}$ be its corresponding eigenvector, then

$$\tilde{\boldsymbol{M}} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \lambda \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix},$$

that is

$$\begin{pmatrix} \boldsymbol{Q}_1 - \boldsymbol{S} & -\boldsymbol{B}^* \\ \boldsymbol{O} & \boldsymbol{Q}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Q_1} + \boldsymbol{P} & \boldsymbol{O} \\ -\boldsymbol{B} & \boldsymbol{Q}_2 \end{pmatrix} \begin{pmatrix} \lambda\boldsymbol{u} \\ \lambda\boldsymbol{v} \end{pmatrix},$$

or

$$\begin{aligned} &[(\lambda - 1)\boldsymbol{Q}_1 + \lambda\boldsymbol{P} + \boldsymbol{S}]\boldsymbol{u} + \boldsymbol{B}^*\boldsymbol{v} = 0 \\ &\lambda\boldsymbol{B}\boldsymbol{u} + (1 - \lambda)\boldsymbol{Q}_2\boldsymbol{v} = 0 \end{aligned}. \tag{2.7}$$

We would use the following Lemmas to proof the convergent of (2.5).

Lemma 2.5 Let $\boldsymbol{A}$ be skew-Hermite positive definite, and $rank(\boldsymbol{B}) = m$, λ be a eigenvalue of $\tilde{\boldsymbol{M}}$,then we have $\lambda \neq 1$.

Proof. If $\lambda = 1$, we have

$$\begin{cases} \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}^*\boldsymbol{v} = 0 \\ \boldsymbol{B}\boldsymbol{u} = 0 \end{cases},$$

or

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^* \\ -\boldsymbol{B} & \boldsymbol{O} \end{pmatrix}\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \boldsymbol{0},$$

but $\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^* \\ -\boldsymbol{B} & \boldsymbol{O} \end{pmatrix}$ is nonsingular, so $\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \boldsymbol{0}$, the result contradicts to the eigenvalue of $\tilde{\boldsymbol{M}}$, so $\lambda \neq 1$.

Lemma 2.6 Let $\boldsymbol{A}$ be skew-Hermite positive definite, $rank(\boldsymbol{B}) = m$, and

$$\boldsymbol{A} = \boldsymbol{P} + \boldsymbol{S},$$

where $\frac{1}{2}(\boldsymbol{P}^* + \boldsymbol{P}) = \frac{1}{2}(\boldsymbol{A}^* + \boldsymbol{A})$, $\boldsymbol{S} := \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^*)$. If $\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix}$ is the corresponding eigenvector of $\tilde{\boldsymbol{M}}$, then

$$\boldsymbol{u} \neq \boldsymbol{0},$$

and when $\boldsymbol{v} = 0$,

$$|\lambda| = \sqrt{\frac{(ad + d^2 - bc)^2 + (ab + bd + cd)^2}{(a+d)^2 + c^2}},$$

where $a + ic = \frac{\boldsymbol{u}^*\boldsymbol{P}\boldsymbol{u}}{\boldsymbol{u}^*\boldsymbol{u}}$, $b = -\frac{\boldsymbol{u}^* i \cdot \boldsymbol{S}\boldsymbol{u}}{\boldsymbol{u}^*\boldsymbol{u}}$, $d = \frac{\boldsymbol{u}^*\boldsymbol{Q}_1\boldsymbol{u}}{\boldsymbol{u}^*\boldsymbol{u}}$, $e = \frac{\boldsymbol{u}^*\boldsymbol{B}^*\boldsymbol{Q}_2^{-1}\boldsymbol{B}\boldsymbol{u}}{\boldsymbol{u}^*\boldsymbol{u}}$.

Proof. If $\boldsymbol{u} = \boldsymbol{0}$, from (2.7), we have

$$\boldsymbol{B}^*\boldsymbol{v} = \boldsymbol{0},$$

where $\boldsymbol{B}^*$ has full row rank, so $\boldsymbol{v} = \boldsymbol{0}$, the result contradicts to the eigenvalue of $\tilde{\boldsymbol{M}}$, so

$$\boldsymbol{u} \neq \boldsymbol{0}.$$

When $\boldsymbol{v} = 0$, from (2.7), we have

$$[(\lambda - 1)\boldsymbol{Q}_1 + \lambda\boldsymbol{P} + \boldsymbol{S}]\boldsymbol{u} = 0,$$

and multiply by $\boldsymbol{u}^*$ on the left, we can get $\lambda(a + ic + d) + ib - d = 0$, so

$$\lambda = \frac{d - bi}{a + ic + d},$$

then

$$|\lambda| = \sqrt{\frac{(ad + d^2 - bc)^2 + (ab + bd + cd)^2}{(a+d)^2 + c^2}}.$$

Theorem 2.1 Let $\boldsymbol{A}$ be skew-Hermite positive definite, and the Hermite part is positive definite, the skew-Hermite part is $\boldsymbol{S} := \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^*)$, $rank(\boldsymbol{B}) = m$, $\boldsymbol{Q}_1 \in \boldsymbol{C}^{n\times n}$ be Hermite matrix, $\boldsymbol{Q}_2 \in \boldsymbol{C}^{m\times m}$ be Hermite positive definite, then (2.5) is convergent if and only if

$$0 < e < \frac{2a^4d^4 - 4a^2b^2c^2d^2 + 3a^3bc + 4a^2c^4d}{(a+d)^2 + c^2}, \tag{2.8}$$

where $a + ic = \frac{\boldsymbol{u}^*\boldsymbol{P}\boldsymbol{u}}{\boldsymbol{u}^*\boldsymbol{u}}$, $b = -\frac{\boldsymbol{u}^* i \cdot \boldsymbol{S}\boldsymbol{u}}{\boldsymbol{u}^*\boldsymbol{u}}$, $d = \frac{\boldsymbol{u}^*\boldsymbol{Q}_1\boldsymbol{u}}{\boldsymbol{u}^*\boldsymbol{u}}$, $e = \frac{\boldsymbol{u}^*\boldsymbol{B}^*\boldsymbol{Q}_2^{-1}\boldsymbol{B}\boldsymbol{u}}{\boldsymbol{u}^*\boldsymbol{u}}$, $\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}$ is the corresponding eigenvector of $\tilde{\boldsymbol{M}}$.

Proof. Let λ be a eigenvalue of $\tilde{\boldsymbol{M}}$, $\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}$ be its corresponding eigenvector. From Lemma2.6, we have

$$\boldsymbol{u} \neq \boldsymbol{0} \text{ and } \lambda \neq 1.$$

From (2.7), we can get

$$[(\lambda-1)\boldsymbol{Q}_1 + \lambda\boldsymbol{P} + \boldsymbol{S}]\boldsymbol{u} - \frac{\lambda}{1-\lambda}\boldsymbol{B}^*\boldsymbol{Q}_2^{-1}\boldsymbol{B}\boldsymbol{u} = 0. \tag{2.9}$$

If $\boldsymbol{B}\boldsymbol{u} = \boldsymbol{0}$, from (2.7) we have

$$[(\lambda-1)\boldsymbol{Q}_1 + \lambda\boldsymbol{P} + \boldsymbol{S}]\boldsymbol{u} = 0.$$

From Lemma 2.6, we can get

$$|\lambda| < 1.$$

If $\boldsymbol{B}\boldsymbol{u} \neq \boldsymbol{0}$,we can get

$$e = \frac{\boldsymbol{u}^*\boldsymbol{B}^*\boldsymbol{Q}_2^{-1}\boldsymbol{B}\boldsymbol{u}}{\boldsymbol{u}^*\boldsymbol{u}} > 0.$$

From (2.9),we have

$$(\lambda-1)d + \lambda(a+ic) + bi - \frac{\lambda}{1-\lambda}e = 0,$$

that is

$$\begin{aligned}&\lambda^2 + \lambda\frac{(ae+de+bc-a^2-ad-c^2-2ad-2d^2)+(ab+bd-ce+cd)i}{(a+d)^2+c^2}\\&+\frac{(ad-bc+d^2)-(ab+bd+cd)i}{(a+d)^2+c^2} = 0\end{aligned} \tag{2.10}$$

From Lemma 2.4, we know $|\lambda| < 1$ if and only if

$$\begin{aligned}&\left|\frac{(ae+de+bc-a^2-ad-c^2-2ad-2d^2)+(ab+bd-ce+cd)i}{(a+d)^2+c^2} - \right.\\&\frac{(ae+de+bc-a^2-ad-c^2-2ad-2d^2)-(ab+bd-ce+cd)i}{(a+d)^2+c^2}\times\\&\left.\frac{(ad-bc+d^2)-(ab+bd+cd)i}{(a+d)^2+c^2}\right| +\\&\left|\frac{(ad-bc+d^2)-(ab+bd+cd)i}{(a+d)^2+c^2}\right|^2 < 1\end{aligned} \tag{2.11}$$

From (2.11),we can get when

$$0 < e < \frac{2a^4d^4 - 4a^2b^2c^2d^2 + 3a^3bc + 4a^2c^4d}{(a+d)^2 + c^2},$$

the iterative scheme is convergent.

NUMERICAL EXPERIMENT

In our numerical experiments, the zero vector as the initial vector, $\begin{pmatrix} \boldsymbol{f}^{\mathrm{T}} \\ \boldsymbol{g}^{\mathrm{T}} \end{pmatrix}$ as the right hand side, then the exact solution of the saddle-point problem is $(\boldsymbol{x}^{\mathrm{T}}, \boldsymbol{y}^{\mathrm{T}})^{\mathrm{T}} = (1,1,\cdots,1)^{\mathrm{T}}$. We use the MATLAB programming experiment, stopping when the error is less than 10^{-5}. IT, CPU and ERR represent the number of iterations, time required for convergence and error respectively.

Example 3.1[17] Consider the following Stokes problem [18]:

$$\begin{cases} -\mu\Delta\mu + \nabla\omega = \mathrm{f}, \quad in \quad \Omega \\ \nabla\cdot \mathrm{u} = \mathrm{g}, \quad in \quad \Omega \\ \mathrm{u} = 0, \quad on \quad \partial\Omega \\ \int_{\Omega} \omega(\mathrm{x})\,\mathrm{dx} = 0, \end{cases}$$

where $\Omega = (0,1)\times(0,1) \subset R^2$ is a square area, $\partial\Omega$ is the boundary condition of Ω, Δ is Laplace operator, vector function u is the speed on Ω,number function ω is pressure. The problem with the upwind difference scheme can be shaped like (1.1), specific form as follows,

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{I}\otimes\boldsymbol{T} + \boldsymbol{T}\otimes\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}\otimes\boldsymbol{T} + \boldsymbol{T}\otimes\boldsymbol{I} \end{pmatrix}, \ \boldsymbol{B} = \begin{pmatrix} \boldsymbol{I}\otimes\boldsymbol{F} \\ \boldsymbol{F}\otimes\boldsymbol{I} \end{pmatrix}$$

where $\boldsymbol{T} = \frac{1}{h^2}\mathrm{tridiag}(-1,2,-1) \in \boldsymbol{R}^{p\times p}$, $\boldsymbol{F} = \frac{1}{h}\mathrm{tridiag}(-1,1,0) \in \boldsymbol{R}^{p\times p}$, $\otimes$ is the Kronecker product of matrix notation, $h = \frac{1}{p+1}$ is the discrete grid value, and $n = 2p^2$, $m = p^2$. Numerical results are given in the following two forms of classical Uzawa method (See Table 3.1) and the new iteration method(See Table3.2).、

Table 3.1 (The Uzawa Method)

n	m	δ	IT	CPU	ERR
50	25	34	19	0.436	7.265e-07
200	100	59	31	1.910	6.839e-06
512	256	120	40	4.219	2.166e-06
1250	625	360	52	98.862	2.109e-06
5000	2500	-	>20000	-	-

Table 3.2 (The New Iteration Method, $\boldsymbol{Q}_1 = \alpha\boldsymbol{I}$ $\boldsymbol{Q}_2 = \frac{1}{\delta}\boldsymbol{I}$ **)**

n	m	α	δ	IT	CPU	ERR
50	25	3	34	14	0.009	3.145e-07
200	100	9	59	23	1.042	2.085e-07
512	256	16	120	32	3.267	1.234e-06
1250	625	55	360	43	79.211	0.337e-06
5000	2500	0.08	-	>20000	-	-

The above numerical experiments, and a large number of not listing of results show that convergence effect of the new iterative algorithm is much better than that of in solving saddle point problems with the classical Uzawa method.

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