Oscillation of the systems of impulsive hyperbolic partial differential equations

Xiaoqi Ning* and Shujun You

Department of Mathematics, Huaihua University, Huaihua, China

ABSTRACT

The systems of impulsive hyperbolic partial differential equations with Robin boundary value condition are investigated. Several new sufficient conditions of oscillation for such systems are established by employing impulsive differential inequalities and integration.

Key words: Impulsive; Delay; The systems of partial differential equations; Oscillation

INTRODUCTION

As an important study field of impulsive partial differential equation, the oscillation theory of impulsive partial differential equation has various applications therefore it has aroused great study interests in recent years, and the study about oscillation theory of impulsive partial differential equations also gradually draws people's attention [1-6]. This paper considers equations

\[
\frac{\partial^2}{\partial t^2} \left[ u_i(t, x) + c(t)u_i(t - \tau, x) \right] + \frac{\partial}{\partial t} \left[ u_i(t, x) + c(t)u_i(t - \tau, x) \right] = a_i(t)\Delta u_i(t, x) + \sum_{j=1}^{s} a_{ij}(t)\Delta u_j(t - \rho_j(t), x) \\
- \sum_{j=1}^{m} p_{ij}(t, x)u_j(t - \sigma_j(t), x), (t, x) \in R \times \Omega,
\]

\[i = 1, 2, \ldots, m; \quad t \neq \tau_k, k = 1, 2, \ldots, i \in I_m = \{1, 2, \ldots, m\}, \]

\[u_i(t_k^-, x) - u_i(t_k^+, x) = \alpha_k u_i(t_k^+, x), \quad k = 1, 2, \ldots, i \in I_m,\]

\[\frac{\partial u_i(t_k^+, x)}{\partial t} - \frac{\partial u_i(t_k^-, x)}{\partial t} = \beta_k \frac{\partial u_i(t_k^+, x)}{\partial t}, \quad k = 1, 2, \ldots, i \in I_m,\]

with boundary condition

\[\frac{\partial u_i(t, x)}{\partial N} + g_i(t, x)u_i(t, x) = 0, \quad x \in \partial \Omega, t \neq t_k, i \in I_m,\]

where, \( R = [0, +\infty), \quad G \equiv R \times \Omega, \quad u_i = u_i(t, x), \quad \Omega \subset R^n \) denotes a bounded domain With piecewise smooth boundary.
\[ \square u_i(t,x) = \sum_{r=1}^{n} \frac{\partial^2 u_r(t,x)}{\partial x_r^2}, \quad (t,x) \in G, \quad i \in I_m. \]

We study the oscillation problem of equation (1) with boundary condition (2), obtain sufficient condition for the oscillation of all solutions and extend the previous results by using existence condition of the eventually positive solution, combined with the impulsive differential inequality. In the following, we will use the conditions as follows:

(H1) \[ 0 < t_1 < t_2 < \cdots < t_k < \cdots, \lim_{k \to +\infty} t_k = +\infty, \quad \lim_{t \to +\sigma(t)} (t - \sigma(t)) = \lim_{t \to -(t - \sigma(t))} = +\infty, \]

\[ 0 \leq \alpha_k \leq \beta_k, \quad k = 1, 2, \cdots, \quad \sigma, \rho, a_r, a_{\infty} \in PC([0, +\infty), [0, +\infty)), \quad c(t) \in C^1(R_+, R_+); \]

(H2) \[ p_j(t,x) \in PC(\overline{G}, R), \quad p_{\infty}(t,x) > 0, \quad p_i(t) = \min_{x \in G} p_i(t,x), \quad \overline{p}_j(t) = \sup_{x \in G} | p_{ij}(t,x) |, \]

\[ Q(t) = \min_{1 \leq i \leq m} \{ p_{\infty}(t) - \sum_{j=1, j \neq i}^{m} \overline{p}_j(t) \} \geq 0, \quad i \in I_m, \quad j \in I_m, \quad \text{PC denotes piecewise smooth continuous function with the following properties: only discontinuous on } t = t_k, \quad k = 1, 2, \cdots \text{ which are discontinuity point of the first kind and left continuous on } t = t_k, \quad k = 1, 2, \cdots; \]

(H3) \[ \text{the } N \text{ in (2) is the unit vector normal to } \partial \Omega, \text{pointing out of } \Omega, \quad g_j(t,x) \text{ is continuous nonnegative function on } [0, +\infty) \times \partial \Omega; \]

(H4) \[ u_i(t,x) \text{ is the solution satisfied the boundary conditions of equation (1), and } \frac{\partial u_i(t,x)}{\partial t} \text{ is the partial derivative of } u_i(t,x) \text{ both are piecewise continuous function with discontinuity point of the first kind on } t = t_k, \quad k = 1, 2, \cdots. \text{Suppose that they satisfy the following equation at impulsion:} \]

\[ u_i(t^-_k, x) = u_i(t_k, x), \quad u_i(t^+_k, x) = (1 + a_k)u_i(t_k, x), \quad k = 1, 2, \cdots, i \in I_m; \]

\[ \frac{\partial u_i(t^-_k, x)}{\partial t} = \frac{\partial u_i(t_k, x)}{\partial t}, \quad \frac{\partial u_i(t^+_k, x)}{\partial t} = (1 + \beta_k) \frac{\partial u_i(t_k, x)}{\partial t}, \quad k = 1, 2, \cdots, i \in I_m. \]

1. THE MAIN RESULT

Theorem 1. Suppose condition (H1)–(H4) satisfied, if second-order impulsive differential inequalities

\[ \begin{cases} w^r(t) + w'(t) + Q(t)w(t - \sigma(t))[1 - c(t - \sigma(t))] \leq 0, \\ t \geq t_0, t \neq t_k, k = 1, 2, \cdots, \\ w(t^-_k) = (1 + \alpha_k)w(t_k), \quad k = 1, 2, \cdots, \\ w^r(t^-_k) \leq (1 + \beta_k)w(t^+_k), \quad k = 1, 2, \cdots, \end{cases} \]

doesn’t have eventually positive solutions, then every non zero solution of problems (1)-(2) is oscillation in domain \( G \).

Proof. Suppose systems (1)-(2) have a nonzero and non-oscillation solution

\[ u(t,x) = (u_1(t,x), u_2(t,x), \cdots, u_m(t,x))^T, \]

let us assume when \( t \geq t_0 \geq 0 \), we have \( |u_i(t,x)| > 0, i = 1, 2, \cdots, m \). Set

\[ \delta_i = \text{sgn} u_i(t,x), \quad z_i(t,x) = \delta_i u_i(t,x), \]

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Then $z_i(t, x) > 0, (t, x) \in [t_0, +\infty) \times \Omega$. From condition (H1) we can see there is $T_i > t_0$, when $t \geq T_i$ we have $z_i(t, x) > 0$, $z_i(t - \rho_i(t), x) > 0$, $(t, x) \in [T_i, +\infty) \times \Omega, i = 1, 2, \ldots, m, l = 1, 2, \ldots, s$.

When $t \neq t_k$ we integrate both sides of equation (1) on $\Omega$

$$
\frac{d^2}{dt^2} \left( \int_\Omega u_i(t, x)dx + c(t) \int_\Omega u_i(t, x)dx \right) + \frac{d}{dt} \left( \int_\Omega u_i(t, x)dx + c(t) \int_\Omega u_i(t, x)dx \right) = a_i(t) \int_\Omega \Delta u_i(t, x)dx + \sum_{j=1}^{m} a_{ij}(t) \int_\Omega \Delta u_i(t - \rho_j(t), x)dx + \\
- \sum_{j=1}^{m} p_{ij}(t) u_j(t - \sigma(t), x)dx, \quad t \geq T_i, \quad i \in I_m, \quad k = 1, 2, \ldots,
$$

(6)

then

$$
\frac{d^2}{dt^2} \left( \int_\Omega z_i(t, x)dx + c(t) \int_\Omega z_i(t, x)dx \right) + \frac{d}{dt} \left( \int_\Omega z_i(t, x)dx + c(t) \int_\Omega z_i(t, x)dx \right) = a_i(t) \int_\Omega \Delta z_i(t, x)dx + \sum_{j=1}^{m} a_{ij}(t) \int_\Omega \Delta z_i(t - \rho_j(t), x)dx + \\
- \sum_{j=1}^{m} \frac{\delta_i}{\delta_j} \int_\Omega p_{ij}(t) z_j(t - \sigma(t), x)dx, \quad t \geq T_i, \quad i \in I_m, \quad k = 1, 2, \ldots.
$$

(7)

It can be derived from Green Theorem and the boundary condition (2) that

$$
\int_\Omega \Delta z_i(t, x)dx = \int_{\partial\Omega} \frac{\partial z_i(t, x)}{\partial N} ds = - \int_{\partial\Omega} g_i(t, x) z_i(t, x)ds \leq 0, \quad t \geq T_i, \quad i \in I_m, \quad k = 1, 2, \ldots,
$$

(8)

$$
\int_\Omega \Delta z_i(t - \rho_j(t), x)dx = \int_{\partial\Omega} \frac{\partial z_i(t - \rho_j(t), x)}{\partial N} ds \\
= - \int_{\partial\Omega} g_i(t - \rho_j(t), x) z_i(t - \rho_j(t), x)ds \leq 0, \quad t \geq T_i, \quad i \in I_m, \quad k = 1, 2, \ldots,
$$

(9)

here $ds$ denotes the area element on $\partial\Omega$. Combining (7),(8),(9) and condition (H2), we can work out the form

$$
\frac{d^2}{dt^2} \left( \int_\Omega z_i(t, x)dx + c(t) \int_\Omega z_i(t, x)dx \right) + \frac{d}{dt} \left( \int_\Omega z_i(t, x)dx + c(t) \int_\Omega z_i(t, x)dx \right) \leq - p_{ij}(t) \int_\Omega z_i(t - \sigma(t), x)dx + \sum_{j=1,j \neq i}^{m} \frac{\omega_i}{\omega_j} \int_\Omega z_j(t - \sigma(t), x)dx, \quad t \geq T_i, \quad i \in I_m, \quad k = 1, 2, \ldots.
$$

(10)

Set

$$
v_i(t) = \int_\Omega z_i(t, x)dx, \quad t \geq T_i, \quad i \in I_m,
$$

By form (10), we have

$$
[v_i(t) + c(t)v_i(t - \tau)]^2 + [v_i(t) + c(t)v_i(t - \tau)]^2 + \\
[\frac{p_{ij}(t)}{\omega_j} v_j(t - \sigma(t))] + \sum_{j=1,j \neq i}^{m} \frac{\omega_i}{\omega_j} [v_j(t - \sigma(t))] \leq 0, \quad t \geq T_i, t \neq t_k, \quad k = 1, 2, \ldots.
$$

(11)
Set

\[ V(t) = \sum_{j=1}^{m} v_j(t), \quad t \geq T_1, \]

Combining this form and form (11), we have

\[ [V(t) + c(t) V(t - \tau)] + [V(t) + c(t) V(t - \tau)] + \]
\[ + \sum_{i=1}^{m} [p_{ti}(t)v_i(t - \sigma(t)) - \sum_{j=1, j \neq i}^{m} \bar{p}_{ij}(t)v_j(t - \sigma(t))] \leq 0, \quad t \geq T_1, k = 1, 2, \ldots \]

and hence

\[ [V(t) + c(t) V(t - \tau)] + [V(t) + c(t) V(t - \tau)] + \]
\[ + \{ \left[ p_{11}(t)v_1(t - \sigma(t)) - \sum_{j=1, j \neq 1}^{m} \bar{p}_{1j}(t)v_j(t - \sigma(t)) \right] + \]
\[ + \left[ p_{22}(t)v_2(t - \sigma(t)) - \sum_{j=1, j \neq 2}^{m} \bar{p}_{2j}(t)v_j(t - \sigma(t)) \right] + \]
\[ + \cdots + \left[ p_{mm}(t)v_m(t - \sigma(t)) - \sum_{j=1, j \neq m}^{m} \bar{p}_{mj}(t)v_j(t - \sigma(t)) \right] \right \} \leq 0, \quad t \geq T_1, k = 1, 2, \ldots \]

furthermore we have

\[ [V(t) + c(t) V(t - \tau)] + [V(t) + c(t) V(t - \tau)] + \]
\[ + \{ \left[ p_{11}(t) - \sum_{j=1, j \neq 1}^{m} \bar{p}_{1j}(t) \right]v_1(t - \sigma(t)) + \left[ p_{22}(t) - \sum_{j=1, j \neq 2}^{m} \bar{p}_{2j}(t) \right]v_2(t - \sigma(t)) + \]
\[ + \cdots + \left[ p_{mm}(t) - \sum_{j=1, j \neq m}^{m} \bar{p}_{mj}(t) \right]v_m(t - \sigma(t)) \} \leq 0, \quad t \geq T_1, k = 1, 2, \ldots \]

then

\[ [V(t) + c(t) V(t - \tau)] + [V(t) + c(t) V(t - \tau)] + \]
\[ + \min_{1 \leq i \leq m} \left\{ p_{ii}(t) - \sum_{j=1, j \neq i}^{m} \bar{p}_{ij}(t) \right\} \sum_{j=1}^{m} v_j(t - \sigma(t)) \leq 0, \quad t \geq T_1, k = 1, 2, \ldots \]

Consequently, we have

\[ [V(t) + c(t) V(t - \tau)] + [V(t) + c(t) V(t - \tau)] + Q(t)V(t - \sigma(t)) \leq 0, \quad t \geq T_1, \]
\[ t \geq T_1, k = 1, 2, \ldots. \]

Set

\[ w(t) = V(t) + c(t)V(t - \tau) \geq V(t) \geq 0, \quad t \geq T_1, \]

then form (13) can be rewrite as

\[ w(t) + w(t) + Q(t)V(t - \sigma(t)) \leq 0, \quad t \geq T_1, k = 1, 2, \ldots, \]

Thus, we have

\[ e^t w(t) + e^t Q(t)V(t - \sigma(t)) \leq 0, \quad t \geq T_1, k = 1, 2, \ldots. \]

From the above discussion, we can see that

\[ w(t) > 0, \quad V(t - \sigma(t)) > 0, \quad e^t w(t) \leq 0, \quad t \neq t_k, k = 1, 2, \ldots, \]

So \( e^t w(t) \) is a monotone decreasing function in \( (t_k, t_{k+1}] \). \( w(t) \) is also a monotone decreasing function in \( (t_k, t_{k+1}] \), it means \( w(t) \leq 0, \quad t \neq t_k, k = 1, 2, \ldots. \)

On the other hand, when \( t = t_k \), by using (3),(4) in \( (H4) \) we have
\[ u_i(t_i^+, x) = (1 + \alpha_i)u_i(t_i^-, x), \quad \frac{\partial u_i(t_i^+, x)}{\partial t} = (1 + \beta_k)\frac{\partial u_i(t_i^-, x)}{\partial t}, \]

Hence

\[ \text{sgn} u_i(t_i^+, x) = \text{sgn} u_i(t_i^-, x), \quad \text{sgn}\left(\frac{\partial u_i(t_i^+, x)}{\partial t}\right) = \text{sgn}\left(\frac{\partial u_i(t_i^-, x)}{\partial t}\right), \]

then

\[ V(t_i^+) - V(t_i^-) = \sum_{i=1}^{m} [v_i(t_i^+) - v_i(t_i^-)] \]

\[ = \sum_{i=1}^{m} \int_{\Omega} (z_i(t_i^+, x) - z_i(t_i^-, x))dx \]

\[ = \sum_{i=1}^{m} \int_{\Omega} \left[ \text{sgn}(u_i(t_i^+, x))u_i(t_i^+, x) - \text{sgn}(u_i(t_i^-, x))u_i(t_i^-, x) \right]dx \]

\[ = \sum_{i=1}^{m} \int_{\Omega} \text{sgn}(u_i(t_i^-, x))[u_i(t_i^+, x) - u_i(t_i^-, x)]dx \]

\[ = \sum_{i=1}^{m} \int_{\Omega} \text{sgn}(u_i(t_i^-, x))\alpha_i u_i(t_i^-, x)dx \]

\[ = \alpha_k V(t_k^-), \]

furthermore, we have

\[ w(t_i^+) - w(t_i^-) = V(t_i^+) + c(t_i^+)V(t_i^- - \tau) - [V(t_i^+) + c(t_i^-)V(t_i^- - \tau)] \]

\[ = V(t_i^+) - V(t_i^-) + c(t_i^-)[V(t_i^+ - \tau) - V(t_i^- - \tau)] \]

\[ = \alpha_k [V(t_i^+) + c(t_i^-)V(t_k^- - \tau)] \]

\[ = \alpha_k \alpha_k(t_k^-), \]

Thus, we have \( w(t_i^+) - w(t_i^-) = \alpha_k w(t_k^-) \). Note that \( w(t_i^-) = w(t_k^-) \), we can also have

\[ w(t_i^+) = (1 + \alpha_k)w(t_k^-). \quad (15) \]

Similarly, we can prove

\[ w'(t_i^+) - w'(t_i^-) \leq \beta_k' w'(t_k). \]

note that

\[ V'(t_i^+) - V'(t_i^-) = \sum_{i=1}^{m} [v_i'(t_i^+) - v_i'(t_i^-)] \]

\[ = \sum_{i=1}^{m} \int_{\Omega} \left[ \frac{\partial z_i(t_i^+, x)}{\partial t} - \frac{\partial z_i(t_i^-, x)}{\partial t} \right]dx \]

\[ = \sum_{i=1}^{m} \int_{\Omega} \left[ \text{sgn}\left(\frac{\partial u_i(t_i^+, x)}{\partial t}\right)\frac{\partial u_i(t_i^+, x)}{\partial t} - \text{sgn}\left(\frac{\partial u_i(t_i^-, x)}{\partial t}\right)\frac{\partial u_i(t_i^-, x)}{\partial t} \right]dx \]

\[ = \sum_{i=1}^{m} \int_{\Omega} \text{sgn}(u_i(t_i^-, x))\left[ \frac{\partial u_i(t_i^+, x)}{\partial t} - \frac{\partial u_i(t_i^-, x)}{\partial t} \right]dx \]

\[ = \sum_{i=1}^{m} \int_{\Omega} \text{sgn}(u_i(t_i^-, x))\beta_k \frac{\partial u_i(t_i^-)}{\partial t} dx \]

\[ = \beta_k \sum_{i=1}^{m} v_i'(t_k) = \beta_k V'(t_k), \]
then

\[ w'(t_k^*) - w'(t_k^-) = [V(t_k^*) + c(t_k^*)V(t_k^* - \tau)] - [V(t_k^-) + c(t_k^-)V(t_k^- - \tau)] \]

\[ = V'(t_k^*) + [c(t_k^*)V(t_k^* - \tau)] - V'(t_k^-) - [c(t_k^-)V(t_k^- - \tau)] \]

\[ = V'(t_k^*) - V'(t_k^-) + c'(t_k^*)V(t_k^* - \tau) + c(t_k^-)V'(t_k^- - \tau) \]

\[ - c'(t_k^-)V(t_k^- - \tau) - c(t_k^-)V(t_k^- - \tau) \]

\[ = [V'(t_k^*) - V'(t_k^-)] + c'(t_k^*)[V(t_k^* - \tau) - V(t_k^- - \tau)] \]

\[ + c(t_k^-)[V'(t_k^- - \tau) - V'(t_k^- - \tau)] \]

\[ = \beta_k V'(t_k^*) + c'(t_k^*)\alpha_k V(t_k^- - \tau) + c(t_k^-)\beta_k V'(t_k^- - \tau) \]

\[ \leq \beta_k [V'(t_k^*) + c'(t_k^*)V(t_k^- - \tau) + c(t_k^-)V'(t_k^- - \tau)] \]

\[ = \beta_k [V(t_k^*) + c(t_k^-)V(t_k^- - \tau)] \]

\[ = \beta_k w'(t_k^-), \]

thus, we have

\[ w'(t_k^*) - w'(t_k^-) \leq \beta_k w'(t_k^-). \]

Furthermore, because \( w'(t_k^-) = w'(t_k^-) \), we have

\[ w'(t_k^*) \leq (1 + \beta_k)w'(t_k^-). \] (16)

By using (15),(16), we can work out

\[ w'(t) \geq 0, \quad t \geq T_1, t \neq t_k. \]

In fact, if the above form is wrong then \( \exists T_2 \geq T_1 \), such that \( w'(T_2) < 0 \), let us assume \( w'(T_2) = -c, \ (c > 0) \). By

\[ w''(t) \leq 0, \quad t \neq t_k, k = 1, 2, \cdots, \quad w'(t_k^*) \leq (1 + \beta_k)w'(t_k^-), \quad t = t_k, k = 1, 2, \cdots, \]

together with lemma 5.1.3 in [6], we have

\[ w'(t) \leq w'(T_2) \prod_{T_2 < t < t_k} (1 + \alpha_k) = -c \prod_{T_2 < t < t_k} (1 + \beta_k), \quad t \geq T_2. \]

Note that \( w'(t_k^*) = (1 + \alpha_k)w(t_k^-) \), then we have

\[ w(t) \leq w(T_2) \prod_{T_2 < t < t_k} (1 + \alpha_k) - c \int_{T_2 < t < t_k} (1 + \alpha_k) \prod_{T_2 < t < s} (1 + \beta_k) ds \]

\[ = \prod_{T_2 < t < t_k} (1 + \alpha_k) \left[ w(T_2) - c \int_{T_2 < t < t_k} (1 + \beta_k) ds \right], \]

note \( 0 < \alpha_k \leq \beta_k \) in (H1), therefore we have \( w(t) \leq 0, \quad t \geq T_2 \), according the above form, this contradicts to \( w(t) > 0, \quad t \geq T_1 \). By \( w(t) = V(t) + c(t)V(t - \tau), \quad t \geq T_1, t \neq t_k, k = 1, 2, \cdots \). together with form (14) we have

\[ w''(t) + w'(t) + Q(t)[w(t - \sigma(t)) - c(t - \sigma(t))V(t - \sigma(t) - \tau)] \leq 0, \]

\[ t \geq T_1, t \neq t_k, k = 1, 2, \cdots. \] (17)

Since

\[ w(t) \geq V(t), \quad w'(t) \geq 0, \quad t \neq t_k, k = 1, 2, \cdots, \]

the form (17) can be rewrite as
therefore combining form (18) and form (15), (16) we can see that \( w(t) \) is a eventually positive solution of differential inequality (5), this contradicts to condition of theorem 1, and we complete the proof of theorem 1.

Theorem 2. Suppose condition \((H1)-(H4)\) satisfied, further more suppose

(i) \( 0 \leq \sigma(t) \leq \sigma, 0 \leq \rho_j(t) \leq \rho, \ j = 1,2,\cdots,m; \)

(ii) if \( \sum_{k=1}^{\infty} \alpha_k < +\infty, \sum_{k=1}^{\infty} \beta_k < +\infty, \) and for each are sufficiently large \( T, \) we have

\[
\lim_{T \to +\infty} \inf_{T < c_l < T} \int_{T} \prod_{T < c_l < T} (1 + \alpha_k) \int_{T} \prod_{T < c_l < T} (1 + \beta_k) \left\{ -Q(s)[1-c(s-\sigma(s))] \right\} dsd\eta \\
\geq -\infty,
\]

Then every nonzero solution of problems (1)-(2) is oscillation in domain G

**Proof.** By theorem 1 we only need to prove the nonexistence of the eventually positive solution of the second order impulsive differential inequality (5). Suppose \( w(t) \) is a eventually positive solution of the impulsive differential inequality (5), then there exists a real number \( T \geq \max \{ \sigma, \rho \}, \) and \( w(T-\sigma) \geq 0, \) in form (5), we notice that \( w(t) \geq 0 \) and \( w(t-\sigma(t)) \geq 0, t \geq T, t \neq t_k, k = 1,2,\cdots, \) hence we have

\[
w''(t) + w'(t) + Q(t)w(t-\sigma(t))[1-c(t-\sigma(t))] \leq 0, t \geq T, t \neq t_k, k = 1,2,\cdots. \tag{18}
\]

combining (19) and \( w'(t_k^+) \leq \left(1 + \beta_k\right)w'(t_k) \), in (5), we can use lemma 5.1.3 in [6] to get

\[
w'(t) \leq w'(T) \prod_{T < c_l < T} (1 + \beta_k) + \\
+ \int_{T} \prod_{T < c_l < T} (1 + \beta_k) \left\{ -w(T-\sigma)Q(s)[1-c(s-\sigma(s))] \right\} ds,
\]

and together with \( w(t_k^+) = (1 + \alpha_k)w(t_k) \) in (5) we have

\[
w(t) \leq w(T) \prod_{T < c_l < T} (1 + \alpha_k) + \int_{T} \prod_{T < c_l < T} (1 + \alpha_k) \left\{ w'(T) \prod_{T < c_l < T} (1 + \beta_k) + \\
+ \int_{T} \prod_{T < c_l < T} (1 + \beta_k) \left\{ -w(T-\sigma)Q(s)[1-c(s-\sigma(s))] \right\} ds \right\} d\eta \\
= w(T) \prod_{T < c_l < T} (1 + \alpha_k) + w'(T) \prod_{T < c_l < T} (1 + \alpha_k) \prod_{T < c_l < T} (1 + \beta_k) d\eta + \\
+ w(T-\sigma) \int_{T} \prod_{T < c_l < T} (1 + \alpha_k) \int_{T} \prod_{T < c_l < T} (1 + \beta_k) \left\{ -Q(s)[1-c(s-\sigma(s))] \right\} ds d\eta
\]

for \( t > T, \) there exists real integer \( m \) such that \( t_m < t \leq t_{m+1}, \) and so
\[
\int_T^T \prod_{T < \eta < \eta} (1 + \beta_k \prod_{\eta < \xi < \eta} (1 + \alpha_k) d\eta
\]
\[
\leq \int_T^T \prod_{T < \eta < \eta} (1 + \alpha_k) d\eta + \sum_{j=1}^{m-1} \int_T^T \prod_{\eta < \xi < \eta} (1 + \beta_k \prod_{\eta < \xi < \eta} (1 + \alpha_k) d\eta + \int_T^T \prod_{T < \eta < \eta} (1 + \beta_k) d\eta
\]
\[
= \prod_{T < \eta < \eta} (1 + \alpha_k) (t_1 - T) + \sum_{j=1}^{m-1} \left[ \int_T^T \prod_{\eta < \xi < \eta} (1 + \beta_k \prod_{\eta < \xi < \eta} (1 + \alpha_k) (t_{j+1} - t_j) \right]
\]
\[
+ \prod_{T < \eta < \eta} (1 + \beta_k) (t_{m+1} - t_m)
\]
\[
< +\infty.
\]
For \( t > T \), the inequality (21) with both sides divided by the left side of (22), we obtain
\[
\frac{\omega(t)}{\prod_{T < \eta < \eta} (1 + \beta_k \prod_{\eta < \xi < \eta} (1 + \alpha_k) d\eta}
\leq \prod_{T < \eta < \eta} (1 + \alpha_k) \omega(T) \quad \omega(T) = \prod_{T < \eta < \eta} (1 + \beta_k \prod_{\eta < \xi < \eta} (1 + \alpha_k) d\eta
\]
\[
+ \omega(T - \sigma) \int_T^T \prod_{\eta < \xi < \eta} (1 + \alpha_k) \prod_{\eta < \xi < \eta} (1 + \beta_k \{-Q(s) [1 - c(s - \sigma(s))] \} ds d\eta
\]
\[
+ \int_T^T \prod_{T < \eta < \eta} (1 + \beta_k \prod_{\eta < \xi < \eta} (1 + \alpha_k) d\eta
\]
\[
\quad \text{Notes}
\]
\[
\int_T^T \prod_{T < \eta < \eta} (1 + \beta_k \prod_{\eta < \xi < \eta} (1 + \alpha_k) d\eta \rightarrow +\infty, \quad t \rightarrow +\infty,
\]
We get
\[
\liminf_{t \rightarrow +\infty} \int_T^T \prod_{T < \eta < \eta} (1 + \beta_k \prod_{\eta < \xi < \eta} (1 + \alpha_k) d\eta = -\infty.
\]
According to condition (ii) in theorem 2, on the other hand by \( w(t) > 0, \quad t \geq T \), we have
\[
\liminf_{t \rightarrow +\infty} \int_T^T \prod_{T < \eta < \eta} (1 + \beta_k \prod_{\eta < \xi < \eta} (1 + \alpha_k) d\eta \geq 0,
\]
This form contradicts to (23), hence impulsive differential equation (5) doesn’t have eventually positive solution, and we complete the proof of theorem 2.

REFERENCES