Available online www.jocpr.com

Journal of Chemical and Pharmaceutical Research, 2014, 6(7):670-678

Research Article

ISSN : 0975-7384
CODEN(USA) : JCPRC5

Geometric equations of arbitrarily spatial curved beam under large deformation

Zeng Sen[1*], Chen Shaofeng[2] and Song Yiran[1]

[1]Department of Civil Engineering, Harbin Institute of Technology at Weihai, China
[2]School of Science and Engineering on Communications, Harbin Institute of Technology, China

---

ABSTRACT

Geometric equations of arbitrarily spatial curved beam include curvature-angle equation, angle-displacement equation and strain-displacement equation. Under the plane cross section assumption, curvature-angle relationship is given out through rigorous mathematical derivation, when arbitrarily spatial curved beam occurs large deformation (including large rotating angle). Based on the above, angle-displacement relationship can be gotten. And finally, Green strain tensor of arbitrarily spatial curved beam under large deformation is presented, which can consider warping. On the basis of the plane cross section assumption, in the case of large rotation and large curvature, the shear deformation of the cross section is caused not only by the torsion angle but also from the other two angles whose contribution is little.

Keywords: Spatial curved beam; Curvature-angle equation; Angle-displacement equation; Green strain tensor; Plane cross section assumption

---

INTRODUCTION

The spatial curved beam is a common structure or component in engineering practices, such as the curved bridge, the arch bridge, etc. It is apparently effective for small curvature curved beams in structural finite element analysis by using a series of straight-beam elements to approximately instead the curved beam, on the contrary, it will highly decrease the calculation efficiency on the premise of guarantee of analysis precision as for large curvature case, and so many scholars are devoted to element analysis of the spatial curved beam. As one of the premises of finite element analysis, the geometric equations for the spatial curved beam must be given, namely the equations of curvature-angle and strain-displacement of three-dimensional deformation for the curved beam. While these relationships are very complicated, they are remarkable characteristics where the curved beam could distinguish from the straight beam, also the emphasis of what scholars study.

Bauchau and Hong [1] have attained creative achievements in this field.Worksby Ojalvo and Newman's[2], Rosen and Rand's [3], Pai and Nayfeh's[4]are ground-breaking. Zhou Weiwen[5] also had analyses systematically.However, these studies are all based on small rotation and small deformation.

Therefore, based on predecessors' successful experience and methods of differential geometry and tensor analysis, this paper gives out equations of curvature-angle, angle-displacement and strain-displacement through rigorous mathematical derivation, when spatial curved beam occurs large deformation (including large rotating angle), where the large rotating angle means trigonometric function won't be expanded as algebraic form in deductions.

Mathematica symbols in this paper follow the rules of dummy index and the rules of free index. Except special points, the value range of every upper and lower index is 1 to 3. In addition, because this paper is described in curve coordinate system, to make clear concept, rigorously distinguishing covariance variables and correspondinginverse variables by positions of the index for the vectors, tensor, and their components, that is, upper indexes corresponds with inverse variables, lower indexes corresponds with covariance variables.

2. THE COORDINATE SYSTEMATIC DESCRIPTION FOR INITIAL STATES OF SPATIAL CURVED BEAM

2.1 THE DESCRIPTION OF ANY POINT ON THE CURVED BEAM

The spatial curved beam can be abstract to a space curve. As showed in Fig. 1, any point $\tilde{P}$ on the axis can use curve length, from starting point of the curve to the point $\tilde{P}$, to make a unique identification. Pick one point O arbitrarily in space, and the sagittal diameter of $\tilde{P}$ is:

$$\tilde{\vec{\boldsymbol{R}}}_0 = \overrightarrow{O\tilde{P}}$$

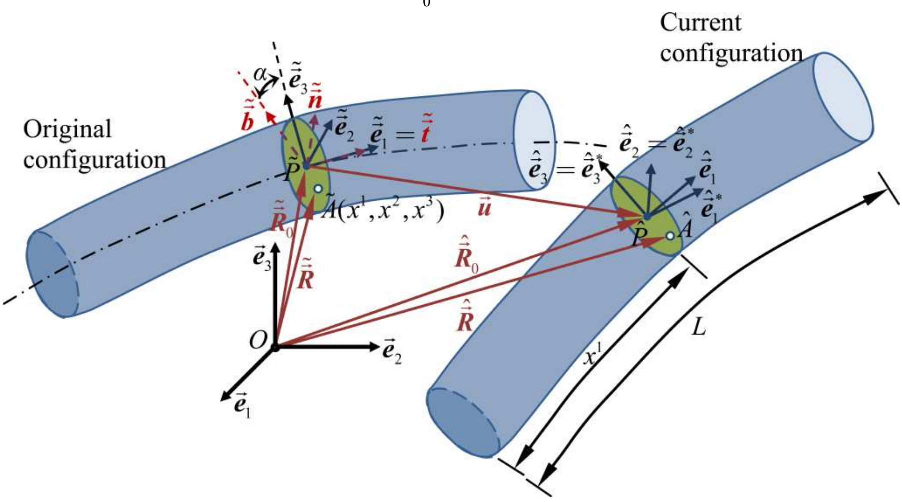

Fig.1Coordinate systems for original and current configuration of spatial curved beam

This paper agrees on boldface variables as tensors and vectors, specially $\vec{\square}$ represented as vectors, and $\tilde{\square}$ represents that the variable is the initial configuration. Apparently, the beam axis is uniquely determined, when the relationship between sagittal diameter and curve lengthx^1 is given.

This paper uses Власов assumption [6], which assumes that the projection of the beam cross sectionkeeps invariant on the original plane in deformationprocesses. It must be pointed out that it is different from the Kirchhoffassumption. The cross section can be made aunique identification based on the position of centroidon the beam axis. To determine the position of the any point $\tilde{A}$ on the section, follow-up coordinate system for central principal axis is built on the section , and three unit orthogonalcovariantbase vectors are $\tilde{\vec{\boldsymbol{e}}}_i$, where $\tilde{\vec{\boldsymbol{e}}}_1$ is along tangential direction of axis, $\tilde{\vec{\boldsymbol{e}}}_2$ and $\tilde{\vec{\boldsymbol{e}}}_3$ are both on the plane of section. Thus, the position and identification of any point $\tilde{A}$ on the section can be determined by three coordinate numbers x^i based on radius vector

$$\overrightarrow{O\tilde{A}} = \tilde{\vec{\boldsymbol{R}}} = \tilde{\vec{\boldsymbol{R}}}_0 + x^2\tilde{\vec{\boldsymbol{e}}}_2 + x^3\tilde{\vec{\boldsymbol{e}}}_3$$

Apparently, when x^2 and x^3 are zero, the point $\tilde{A}$ is the point $\tilde{P}$.

Additionally, according to the knowledge of differential geometry, axis tangentialbase vector $\tilde{\vec{\boldsymbol{e}}}_1$ is equal to the differential coefficient of $\tilde{\vec{\boldsymbol{R}}}_0$ with respect to x^1

$$\tilde{\vec{\boldsymbol{e}}}_1 = \tilde{\vec{\boldsymbol{R}}}_{0,1}$$

where $\square_{,i}$ represents the differential coefficient of the variable with respect to x^i.

2.2 THE DIFFERENTIAL COEFFICIENT OF $\tilde{\vec{e}}_i$ WITH RESPECT TO X^1

The $\tilde{\vec{e}}_i$ which change as the coordinate x^1 are variable. The differential coefficient of $\tilde{\vec{e}}_i$ with respect to x^1 should be calculated for the following analyses, which could be attained based on Frenet-Serret formula [7]·

According to differential geometry, there is a following relationship among unit tangential vector $\tilde{\vec{t}}$, unit principal normal vector $\tilde{\vec{n}}$, unit binormal vector $\tilde{\vec{b}}$ vector, and the differential coefficient of eachof them with respect to x^1:

$$\begin{Bmatrix} \tilde{\vec{t}}_{,1} \\ \tilde{\vec{n}}_{,1} \\ \tilde{\vec{b}}_{,1} \end{Bmatrix} = \begin{bmatrix} 0 & \tilde{\kappa} & 0 \\ -\tilde{\kappa} & 0 & \tilde{\upsilon} \\ 0 & -\tilde{\upsilon} & 0 \end{bmatrix} \begin{Bmatrix} \tilde{\vec{t}} \\ \tilde{\vec{n}} \\ \tilde{\vec{b}} \end{Bmatrix}$$

where $\tilde{\kappa}$ is the curvature, $\tilde{\upsilon}$ is the torsion. Asshowed in Fig. 1, $\tilde{\vec{e}}_1$ is $\tilde{\vec{t}}$. While $\tilde{\vec{e}}_2$, $\tilde{\vec{e}}_3$ and $\tilde{\vec{n}}$, $\tilde{\vec{b}}$ are generally of misalignment, there is a following relationship between two groups of the unit vector .(α is the angle between $\tilde{\vec{e}}_2$ and $\vec{n}$.)

$$\begin{Bmatrix} \tilde{\vec{e}}_1 \\ \tilde{\vec{e}}_2 \\ \tilde{\vec{e}}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} \tilde{\vec{t}} \\ \tilde{\vec{n}} \\ \tilde{\vec{b}} \end{Bmatrix}$$

Thereuponthe differential coefficient of $\tilde{\vec{e}}_i$ with respect to x^1 can be obtained.

$$\tilde{\vec{e}}_{i,1} = \tilde{K}_i^j \tilde{\vec{e}}_j ;\ [\tilde{K}_i^j] = \begin{bmatrix} 0 & \tilde{\kappa}_3 & -\tilde{\kappa}_2 \\ -\tilde{\kappa}_3 & 0 & \tilde{\kappa}_1 \\ \tilde{\kappa}_2 & -\tilde{\kappa}_1 & 0 \end{bmatrix} \quad (1)$$

$$\tilde{\kappa}_1 = \tilde{\upsilon} + \alpha_{,1};\quad \tilde{\kappa}_2 = \tilde{\kappa}\sin\alpha;\quad \tilde{\kappa}_3 = \tilde{\kappa}\cos\alpha$$

$[\square_i^j]$, $[\square^{ij}]$, $[\square_{ij}]$ and $[\square]$ represent matrices in this paper (the matrices corresponding to second-order tensor are also represented so) , and $\square_i^j$, $\square^{ij}$ and $\square_{ij}$ represent the component of i row j column or tensorial component for the matrix. The derivation of $\tilde{\vec{e}}_i$ is conveniently transformed to onelinear combination by formula(1).

3. THE COORDINATE SYSTEMATIC DESCRIPTION FOR CURRENT STATES OF SPATIAL CURVED BEAM

3.1 THE DESCRIPTION OF ANY POINT FOR CURRENT STATES ON THE CURVED BEAM

Former point $\tilde{P}$ on the axis will take place a displacement and move to point $\hat{P}$, when the curved beamproduces deformation. As showed in Fig. 1, here the sagittal diameter of point $\hat{P}$ can be obtained by formal sagittal diameter adding a displacement vector.

$$\hat{\vec{R}}_0 = \tilde{\vec{R}}_0 + \vec{u}$$

To emphasize once more, the $\hat{\square}$ represents the vector of current states.

The displacement vector $\vec{u}$ is unfolded to its component based on $\tilde{\vec{e}}_i$, thus,

$$\hat{\vec{R}}_0 = \tilde{\vec{R}}_0 + u^i \tilde{\vec{e}}_i ; \qquad \hat{\vec{e}}_1 = \hat{\vec{R}}_{0,1}$$

Similar to the initial states, concomitant coordinate systemfor central principal axis is built on the section for current states ,and threecovariantbase vectors are $\hat{\vec{e}}_i$ .Note that because of effects of axial strain and shear strain, $\hat{\vec{e}}_1$ is neither an unit vector nor orthogonal to $\hat{\vec{e}}_2$ and $\hat{\vec{e}}_3$, but $\hat{\vec{e}}_2$ and $\hat{\vec{e}}_3$ are still a pair of orthogonal unit vectors based on

Власов assumption. Here another group of unit orthogonalcovariantbase vectorsis introduced, to be convenient to follow description.

$$\hat{\vec{e}}_1^* = \hat{\vec{e}}_2 \times \hat{\vec{e}}_3; \quad \hat{\vec{e}}_2^* = \hat{\vec{e}}_2; \quad \hat{\vec{e}}_3^* = \hat{\vec{e}}_3$$

The point is $\hat{A}$ in the current state when the initial state point $\tilde{A}$ takes place displacement. Because of Власов assumption, on the coordinate of the base vectors $\hat{\vec{e}}_i^*$ for current states, there is

$$\begin{aligned} \overrightarrow{O\hat{A}} &= \hat{\vec{R}} = \hat{\vec{R}}_0 + x^2\hat{\vec{e}}_2 + x^3\hat{\vec{e}}_3 = \hat{\vec{R}}_0 + x^2\hat{\vec{e}}_2^{\,*} + x^3\hat{\vec{e}}_3^{\,*} \\ &= \tilde{\vec{R}}_0 + \vec{u} + x^2\hat{\vec{e}}_2^{\,*} + x^3\hat{\vec{e}}_3^{\,*} \end{aligned}$$

3.2 THE DIFFERENTIAL COEFFICIENT OF $\tilde{\vec{e}}_i$ WITH RESPECT TO X^1 BASED ON KIRCHHOFF ASSUMPTION

Generally, $\hat{\vec{e}}_1$ is neither a unit vector nor orthogonal to $\hat{\vec{e}}_2$ and $\hat{\vec{e}}_3$, because of effects of axial strain and shear strain. And because the current states $\hat{\vec{n}}$ and $\hat{\vec{b}}$ are non-coplanar with $\hat{\vec{e}}_2$ and $\hat{\vec{e}}_3$, which causes there is no inevitable relationship among $\hat{\vec{e}}_i^*$ and current states $\hat{\vec{t}}$, $\hat{\vec{n}}$ and $\hat{\vec{b}}$, the derivative relations similar to the formula (1) cannot be obtained under the Власов assumption.

The derivation of $\hat{\vec{e}}_i^*$ can be transformed to onelinear combination similar to formula (1), which could copy the treatment of initial state, only when the Kirchhoff assumption is introduced:

$$\hat{\vec{e}}_{i,1}^* = \hat{K}_i^{\,j}\hat{\vec{e}}_j^* \tag{2}$$

where specific componentsof $\hat{K}_i^{\,j}$ are same as the form of formula (1), as long as to replace initial state item $\tilde{\square}$ in theformula (1) with the current state item $\hat{\square}$.

3.3 COORDINATE CONVERSIONS OF INITIAL AND CURRENT STATES: CONVERSIONS OF $\tilde{\vec{e}}_i$ AND $\hat{\vec{e}}_i^*$

Because $\tilde{\vec{e}}_i$ and $\hat{\vec{e}}_i^*$ are both unit orthogonalcovariantbase vectors, conversions between them can always be finished based on a following form.

$$\hat{\vec{e}}_i^* = T_i^{\,j}\tilde{\vec{e}}_j$$

where $[T_i^{\,j}]$, which could be obtained by revolving around the axis through angle θ^i three times,independence among three rotation angles must be ensured, is the matrix for coordinate transformation.

There are two approaches to obtain $[T_i^{\,j}]$. One is called the Cardan angle method[8].Namely θ^i is the rotation angle revolving about real-time axis $\hat{\vec{e}}_i^{\,*}{}'$, where $\hat{\vec{e}}_i^{\,*}{}'$ is the axis that coordinate system revolves about every time as saw in Fig. 2.

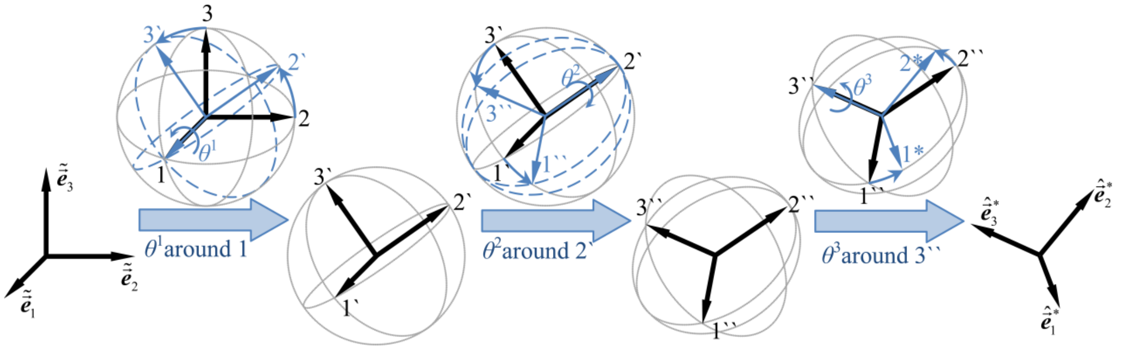

Fig.2 Cardan angle method for coordinatetransformation

While the position of a point on the bar and its cross section could be determined by three displacements and three angular displacements. On the condition of making sure independence among three rotation angles, three angular displacements θ^icould be regarded as the component of angular vectors based on $\tilde{\tilde{e}}_i$ by comparing with the component of $\vec{u}$ unfolded based on $\tilde{\tilde{e}}_i$, which is also corresponding to Green strain mentioned in following paper. This paper doesn't adopt Cardau angle method, because the three angular displacements θ^i in Cardau angle method are based on $\hat{\tilde{e}}_i^{*}\,'$.

According to above thinking, $\tilde{\tilde{e}}_i$ in the order of i=1,2,3 revolves around a fixed axis $\tilde{\tilde{e}}_i\,'$showed in Fig.3 through angle θ^i to obtain $\tilde{\tilde{e}}_i$. The main contents are as follows: firstly, $\tilde{\tilde{e}}_i$ revolves around $\tilde{\tilde{e}}_1\,'$through angle θ^1 according to right-hand rule to obtain the new coordinate axis $\tilde{\tilde{E}}_i$, and $\tilde{\tilde{E}}_i$ is expressed as the component $\tilde{\tilde{E}}_i\tilde{\tilde{e}}_i\,'$ based on fixed axis $\tilde{\tilde{e}}_i\,'$. Then to obtain the expression of the new coordinate axis $\tilde{\tilde{E}}_i'$, the coordinate axis $\tilde{\tilde{E}}_i$ got from the previous step, revolves aroundthe fixed axis $\tilde{\tilde{e}}_2\,'$through angle θ^2according to right-hand rule, which is equivalent to the corresponding components $\tilde{\tilde{E}}_i\tilde{\tilde{e}}_i\,'$ revolve around $\tilde{\tilde{e}}_2\,'$ through angle θ^2. Finally $\tilde{\tilde{E}}_i'$ revolves around fixed axis $\tilde{\tilde{e}}_3\,'$ through angle θ^3, which is equivalent that the component $\tilde{\tilde{E}}_i\,'\tilde{\tilde{e}}_i\,'$ of $\tilde{\tilde{E}}_i'$ based on fixed axis $\tilde{\tilde{e}}_i\,'$ revolves around $\tilde{\tilde{e}}_3\,'$ through angle θ^3, to obtain final transformationmatrix $[T_i^{\,j}]$. It's important to note that to make sure the cross section of a curved beam are still plane after deformation, the rotation order of coordinate axis should be $\tilde{\tilde{e}}_1\,'\rightarrow\tilde{\tilde{e}}_2\,'\rightarrow\tilde{\tilde{e}}_3\,'$.

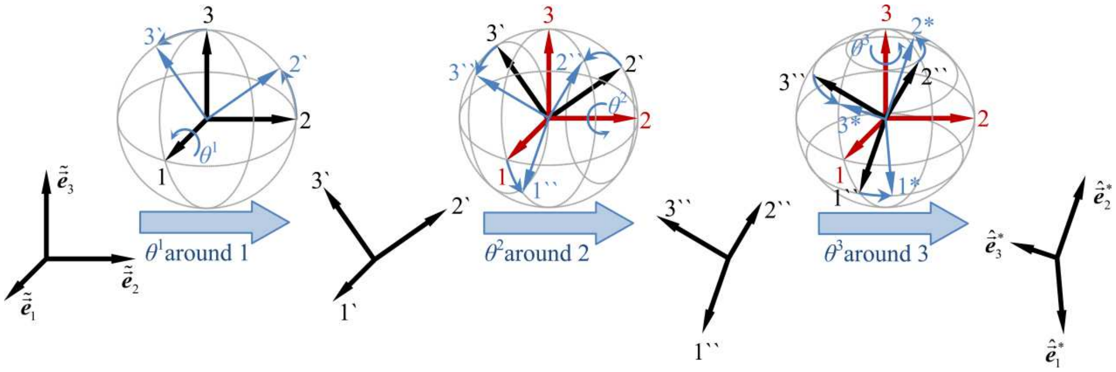

Fig.3 Coordinate transformation method in this paper

The concrete form of the transformation matrix is

$$[T_i^{\,j}]=\begin{bmatrix} c_2c_3 & c_2s_3 & -s_2 \\ s_1s_2c_3-c_1s_3 & s_1s_2s_3+c_1c_3 & s_1c_2 \\ c_1s_2c_3+s_1s_3 & c_1s_2s_3-s_1c_3 & c_1c_2 \end{bmatrix}$$

$$c_i=\cos\theta^i;\quad s_i=\sin\theta^i$$

also can be written as

$$[T_i^{\,j}]=[A_1][A_2][A_3]$$

$$[A_1]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & -s_1 & c_1 \end{bmatrix};[A_2]=\begin{bmatrix} c_2 & 0 & -s_2 \\ 0 & 1 & 0 \\ s_2 & 0 & c_2 \end{bmatrix};[A_3]=\begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is clear that $[A_i]$ is orthotropic and only relevant to angle θ^i, which proofs $[T_i^{\,j}]$ is also orthotropic.

In addition, we can find that the transformation matrix obtained according to revolve around real-time axis 3, 2, 1 through angle $\theta^i$is formally equivalent to $[T_i^{\,j}]$ in this paper. Please note that meaning of angle $\theta^i$is different in Cardau angle method.

4. THE CURVATURE-ANGLE-DISPLACEMENT RELATIONSHIP OF SPATIAL CURVED BEAM UNDER THE KIRCHHOFF ASSUMPTION.

4.1 THE CURVATURE-ANGLE RELATIONSHIP UNDER THE KIRCHHOFF ASSUMPTION.

Under the Власов assumption, a plane may be non-planar and wrapped after deformation, so in this case there is no curvature-angle-displacement relationship. The result in literature[5]is based on the Kirchhoff assumption. The curvature-angle relationship based on the Kirchhoff assumption is as followed.

According to formula (1~3) and considering the orthogonality of [T_i^j], there is

$$\hat{K}_i^j = T_i^k \tilde{K}_k^l T_l^j + T_{i,1}^l T_l^j$$

Then we can get

$$\begin{Bmatrix} \hat{\kappa}_1 \\ \hat{\kappa}_2 \\ \hat{\kappa}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -s_2 \\ 0 & c_1 & s_1c_2 \\ 0 & -s_1 & c_1c_2 \end{bmatrix} \begin{Bmatrix} \theta_{,1}^1 \\ \theta_{,1}^2 \\ \theta_{,1}^3 \end{Bmatrix} + [T_i^j] \begin{Bmatrix} \tilde{\kappa}_1 \\ \tilde{\kappa}_2 \\ \tilde{\kappa}_3 \end{Bmatrix} \quad (4)$$

This is the curvature-angle relationship of spatial curved beam. In the derivation process, thetrigonometric functionhas not been reduced to the angle, so this result is suitable for large angle case. And after getting the sectional angle, the curvature of the current state will be calculated according to formula(4).

When considering the issue of small curvature and small rotation, we think that $\sin\theta^i \approx 0$, $\cos\theta^i \approx 1$. So the results of formula (4) and literature [5] are the same. Moreover, when the value of the initial curvature is zero, formula (4) is reduction to the curvature-angle relationship for straight beam: the curvature is equal to the derivative of angle with respect to x^1.

5. THE TRAIN-DISPLACEMENT RELATIONSHIP OF SPATIAL CURVED BEAM

5.1 THE STRAIN-DISPLACEMENT RELATIONSHIP UNDER GENERAL CONDITION

In order to describe the strain of any point $\tilde{A}$ on section, as shown in Fig. 1, the curve coordinate system is established on point $\tilde{A}$, and three concomitant base vectors are $\tilde{\vec{g}}_i$, this base vector is neither unit vector nor orthogonal.

According to tensor analysis and formula(1), we can get

$$\tilde{\vec{g}}_1 = \tilde{\vec{R}}_{,1} = \tilde{\vec{e}}_1 + (x^2 \tilde{K}_2^j + x^3 \tilde{K}_3^j)\tilde{\vec{e}}_j$$

$$\tilde{\vec{g}}_2 = \tilde{\vec{R}}_{,2}(x^i) = \tilde{\vec{e}}_2 \,; \quad \tilde{\vec{g}}_3 = \tilde{\vec{R}}_{,3}(x^i) = \tilde{\vec{e}}_3$$

So the covariant component of metric tensor $\tilde{\boldsymbol{g}}$ of point $\tilde{A}$ in curve coordinate system is

$$\tilde{g}_{ij} = \tilde{\vec{g}}_i \cdot \tilde{\vec{g}}_j$$

And its matrix form is

$$[\tilde{g}_{ij}] = \begin{bmatrix} g + (x^3\tilde{\kappa}_1)(x^3\tilde{\kappa}_1) + (x^2\tilde{\kappa}_1)(x^2\tilde{\kappa}_1) & -x^3\tilde{\kappa}_1 & x^2\tilde{\kappa}_1 \\ & 1 & \\ \text{sym.} & & 1 \end{bmatrix}$$

$$g = (1 - x^2\tilde{\kappa}_3 + x^3\tilde{\kappa}_2)^2$$

Likewise, it can be inferred that the concomitant base vectors $\hat{\vec{g}}_i$ in curve coordinate system of current state point $\hat{A}$ is

$$\hat{\vec{g}}_1 = \hat{\vec{R}}_{,1}(x^i) = \tilde{\vec{e}}_1 + \alpha^i \tilde{\vec{e}}_i$$

$$\hat{\vec{g}}_2 = \hat{\vec{R}}_{,2} = T_2^i \tilde{\vec{e}}_i \,; \quad \hat{\vec{g}}_3 = \hat{\vec{R}}_{,3} = T_2^i \tilde{\vec{e}}_i$$

$$\alpha^i = u_{,1}^i + u^j \tilde{K}_j^i + \beta_{,1}^i + \beta^j \tilde{K}_j^i \,; \qquad \beta^i = x^2 T_2^i + x^3 T_3^i$$

The covariant component of metric tensor $\tilde{\boldsymbol{g}}$ is

$$\hat{g}_{ij} = \hat{\tilde{\boldsymbol{g}}}_i \cdot \hat{\tilde{\boldsymbol{g}}}_j$$

And its matrix form is

$$[\hat{g}_{ij}] = \begin{bmatrix} 1+2\alpha^1+\alpha^i\alpha^i & T_2^1+\alpha^i T_2^i & T_3^1+\alpha^i T_3^i \\ & 1 & 0 \\ \text{sym.} & & 1 \end{bmatrix}$$

So we can get the strain tensor of point $\tilde{A}$ according to the definition formula of Green strain

$$\boldsymbol{E} = E_{ij}\tilde{\tilde{\boldsymbol{g}}}^i\tilde{\tilde{\boldsymbol{g}}}^j = \frac{1}{2}(\hat{g}_{ij} - \tilde{g}_{ij})\tilde{\tilde{\boldsymbol{g}}}^i\tilde{\tilde{\boldsymbol{g}}}^j \quad (7)$$

$$2E_{11} = 1+2\alpha^1+\alpha^k\alpha^k - g - (x^3\tilde{\kappa}_1)(x^3\tilde{\kappa}_1) - (x^2\tilde{\kappa}_1)(x^2\tilde{\kappa}_1)$$
$$2E_{12} = 2E_{21} = T_2^1 + \alpha^k T_2^k + x^3\tilde{\kappa}_1$$
$$2E_{13} = 2E_{31} = T_3^1 + \alpha^k T_3^k - x^2\tilde{\kappa}_1$$
$$E_{22} = E_{33} = E_{23} = E_{32} = 0$$

Formula (7) gives the strain tensor $\boldsymbol{E}$and the covariant component which are corresponding to the contravariant base $\tilde{\tilde{\boldsymbol{g}}}^i$ of $\tilde{\tilde{\boldsymbol{g}}}_i$. It can be inferred from formula(7) that according toВласов assumption, E_{22}=E_{33}=E_{23}=0, only do axial strain component and the shear strain component exists on the cross section. In addition, from $E_{11,}$ it can be seen that the relationship in the formula(7) could consider the section warping.

For convenient application, the contra variantcomponent corresponding to tensor $\boldsymbol{E}$and $\tilde{\tilde{\boldsymbol{e}}}_i$ should be given.Because of

$$\boldsymbol{E} = e^{ij}\tilde{\tilde{\boldsymbol{e}}}_i\tilde{\tilde{\boldsymbol{e}}}_j = e^{ij}Q_{ik}Q_{jl}\tilde{\tilde{\boldsymbol{g}}}^k\tilde{\tilde{\boldsymbol{g}}}^l = E_{kl}\tilde{\tilde{\boldsymbol{g}}}^k\tilde{\tilde{\boldsymbol{g}}}^l$$

Therefore

$$e^{ij}Q_{ik}Q_{jl} = E_{kl}; \quad Q_{ij} = \tilde{\tilde{\boldsymbol{e}}}_i \cdot \tilde{\tilde{\boldsymbol{g}}}_j$$

So

$$ge^{11} = E_{11} + 2x^3\tilde{\kappa}_1 E_{12} - 2x^2\tilde{\kappa}_1 E_{13}$$
$$\sqrt{g}e^{12} = \sqrt{g}e_{21} = E_{12}; \quad \sqrt{g}e^{13} = \sqrt{g}e_{31} = E_{13} \quad (8)$$
$$e^{22} = e^{33} = e^{23} = e^{32} = 0$$

This is the angle-displacement relationship of large angle and large deformation for spatial curved beam under the general condition.

5.2 THE DISCUSSION ABOUT THAT KIRCHHOFF ASSUMPTION IS EQUAL TO NO SHEAR DEFORMATION.

In the previous paper, it has concluded that Kirchhoff assumption in formula(5) is necessary to get the relationship between angle and displacement. It will be proved in the following part that Kirchhoff assumption is equivalent to no shear deformation in terms of a small rotation and small curvature.

Through observation of vector $\hat{\tilde{\boldsymbol{g}}}_1$ it can be found that

$$\alpha^1 = B^1 + \beta^1_{,1} + \beta^j\tilde{K}^1_j - 1$$
$$\alpha^i = B^i + \beta^i_{,1} + \beta^j\tilde{K}^i_j; \quad (i = 2,3)$$

Then, the shear strain on the cross section can be rewrite to

$$2E_{12} = B^iT_2^i - x^3\theta^1_{,1} + x^3\theta^3_{,1}s_2 - x^3T_1^i\tilde{\kappa}_i + x^3\tilde{\kappa}_1$$
$$2E_{13} = B^iT_3^i + x^2\theta^1_{,1} - x^2\theta^3_{,1}s_2 + x^2T_1^i\tilde{\kappa}_i - x^2\tilde{\kappa}_1$$

If formula (5) is tenable, for the problem of small rotation and small curvature, the shear strain is

$$2E_{12} = -x^3\theta^1_{,1}; \qquad 2E_{13} = x^2\theta^1_{,1}$$

So the shear strain is only caused by torsion angle $\theta^{1.}$ And Kirchhoff assumption, that the issue of small rotation and small curvature is equivalent to no shear deformation, is proved.

From above we can also get, for the large rotation and large curvature, after considering the Kirchhoff assumption, the shear strain of the cross section is not only caused by torsion angleθ^1, and also by the other two angles, but their contribution is little.

At last, the other form of axial strain is

$$\begin{aligned} E_{11} = & \frac{1}{2}B^iB^i + (B^i\beta^i)_{,1} - \beta^iB^i_{,1} + B^i\beta^j\tilde{K}^i_j \\ & + \frac{1}{2}\beta^i_{,1}\beta^i_{,1} + \beta^j\tilde{K}^i_j\beta^i_{,1} + \frac{1}{2}\beta^j\tilde{K}^i_j\beta^k\tilde{K}^i_k \\ & - \frac{1}{2}g - \frac{1}{2}(x^3\tilde{\kappa}_1)(x^3\tilde{\kappa}_1) - \frac{1}{2}(x^2\tilde{\kappa}_1)(x^3\tilde{\kappa}_1) \end{aligned}$$

So it can be known that based on Власов assumption, it is a well-known conclusion that restrained torsion or warping will happen in curved beams. And after the introducing of Kirchhoff assumption, the second term of the upper formula is equal to zero.

CONCLUSION

Conclusion:
(1)Through rigorous mathematical derivation, this article has obtained that, under the Kirchhoff assumption, the curvature—angle relationship of large angle and large deformation in any spatialcurved beam，and further get the angle—displacement relationship.

(2) This paper gives the component, which is in curve local coordinate system and section central principal axis coordinate system, of Green strain tensor of large angle and large deformation in any curved beam in general condition.

(3) It also proves that Kirchhoff assumption is equivalent to no shear deformation in terms of small rotation and small curvature; but for the large rotation and large curvature, after considering the Kirchhoff assumption, the shear strain in the cross section is not only caused by torsion angleθ^1, and also by the other two angles, but their contribution is little.

Outlook:
(1) Although this paper gives the geometric deformation relationship of large angle and large deformation in any spatial curved beam, we can find in the discussion that many of these relationships are transcendental equation of trigonometric function containing angle, which is the inevitable result of considering large rotation. And it is not convenient to establish the finite element formulation, so it needs to simplify these relationships to trigonometric functional algebraic equation of large limited rotation not containing angle.

(2) This paper has not specified the shape of curved beams, in fact the forms of curved beams in terms of engineering is limited, so we may could further according to some common shapes of spatial curved beam members, and use several geometric deformation relationships given in this paper to build up particular curved beam element.

Acknowledgments
The research reported herein was supported by the Scientific Research Foundation of Harbin Institute of Technology at Weihai, and the National Natural Science Foundation of China (Grant No. 51308157). The results, opinions, and conclusions expressed in this paper are solely those of the writers and do not necessarily represent those of the sponsors.

REFERENCES

[1] Bauchau O A, Hong C H. *Journal of Applied Mechanics* 55(match), 156~163, **1988**
[2] Ojalvo I U, Newman M. *Journal of the Engineering Mechanics Division, ASCE* 94(EM5), 1067~1087, **1968**

[3] Rosen A, Rand O. *Compt Struct* 22(5), 785~799, **1986**
[4] Pai P F, Nayfeh A H. *Int J Solids Structures* 31(9), 1309~1340, **1994**
[5] CHEN Da-peng, ZHOU Wen-wei. *Journal of Southwest Jiaotong University* 32(2), 123~129,**1997**
[6] ВЛАСОВ В З.*Тонкостенные упрутие стержни*, Иэд2-е Фиэматгиэ ,**1959**
[7] SUN Bu-qing, HU He-sheng, SHEN Chun-li et.Differential Geometry. *Higher Education Press*, Beijing,**1979**
[8] YUAN Shi-jie, LU Zhe-qin. Multi-rigid-body system dynamics. *Beijing Institute of Technology Press*, Beijing, **1992**